Euclid's "Elements" Redux

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EUCLID'S "ELEMENTS" REDUX

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Email the editor with questions, comments, corrections, and additions: editor@starrhorse.com

"Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?"

- Paul Halmos

"Pure mathematics is, in its way, the poetry of logical ideas."

- Albert Einstein

FIGURE 0.0.1. Extreme close-up of a snowflake.

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Part 1

Introduction

CHAPTER 1

About this project

The goal of this textbook is to provide an open, low-cost, readable edition of Euclid's "Elements" that can be distributed anywhere in the world. (In terms of the American educational system, this textbook may be used in grades 7-12 as well as undergraduate college courses on proof writing).

Euclid's "Elements" was the foremost math textbook in most of the world for about 2,200 years. Many problem solvers throughout history wrestled with Euclid as part of their early education including Copernicus, Kepler, Galileo, Sir Isaac Newton, Abraham Lincoln, Bertrand Russell, and Albert Einstein.

However, "The Elements" was abandoned after the explosion of new mathematics toward the end of the 19th century, including the construction of formal logic, a more rigorous approach to proof-writing, and the necessity of algebra as a prerequisite to calculus. While the end of the 19th century was the beginning of a mathematical Golden Age (one that we are still in), many considered Euclid to be hopelessly out of date.

Should "The Elements" be sufficiently rewritten to conform to the current textbook standards, its importance in geometry, proof writing, and as a case-study in the use of logic may once again be recognized by the worldwide educational community.

This is a goal that no one author can accomplish. As such, this edition of Euclid has been released under the Creative Commons Attribution-ShareAlike 4.0 International License. The intent is to take advantage of crowd-sourcing in order to improve this document in as many ways as possible.

"Euclid's 'Elements' Redux" began as "The First Six Books of the Elements of Euclid" by John Casey (which can be downloaded from Project Gutenberg: `http://www.gutenberg.org/ebooks/21076`), the public domain translation of "The Elements" by Sir Thomas L. Heath, information from Wikipedia and other sources with appropriate licensing, and it includes illustrations composed on GeoGebra software as well as original writing.

The ultimate goal for this document is to contain all 13 books of "The Elements" (some perhaps in truncated form) and to be translated as many languages as possible. Some may also wish to fork this project in order to rewrite

Euclid from the ground up, to create a "purist's" edition (the current edition favors Casey's amendments to Euclid's original work), to create a wiki of mathematics from the ancient world, or for some other reason. Such efforts are welcome.

The prerequisites for this textbook include a desire to solve problems and to learn mathematical logic. Some algebra will be helpful, especially proportions.

This textbook requires its student to work slowly and carefully through each section. The student should check every result stated in the book and not take anything on faith. While this may seem tedious, it is exactly this attention to detail which separates those who understand mathematics from those who do not.

The figures in this textbook were created in GeoGebra. They can be found in the images folder in the source files for this textbook. Files with extensions .ggb are GeoGebra files, and files with the .eps extension are graphics files. Instructional videos are also available on YouTube which demonstrate how to use GeoGebra: `http://www.youtube.com/channel/UCjrVV46Fijv-Pi5VcFm3dCQ`

This document was composed using:

GeoGebra	`http://www.geogebra.org/`
Linux Mint	`http://www.linuxmint.com/`
LyX	`http://www.lyx.org/`
Windows 7	`http://windows.microsoft.com/`
Xubuntu Linux	`http://xubuntu.org/`

Follow me on Twitter: @euclidredux

1.1. Contributors & Acknowledgments

- Daniel Callahan (general editor)
- Deirdre Callahan (cover art using GIMP on Raphael's "The School of Athens")
- John Casey (his edition of "The Elements" is this basis for this edition).
- Sir Thomas L. Heath (various proofs)

Daniel Callahan would like to thank Wally Axmann[1], Elizabeth Behrman[2], Karl Elder[3], Thalia Jeffres[4], Kirk Lancaster[5], Phil Parker[6], and Weatherford College[7] for time and facilities to work on this project.

1.2. Dedication

This book is dedicated to everyone in the educational community who believes that algebra provides a better introduction to mathematics than geometry.

[1]http://www.math.wichita.edu/~axmann/
[2]http://webs.wichita.edu/physics/behrman/behr.htm
[3]http://karlelder.com
[4]http://www.math.wichita.edu/~jeffres/
[5]http://kirk.math.wichita.edu/
[6]http://www.math.wichita.edu/~pparker/
[7]http://www.wc.edu

CHAPTER 2

About Euclid's "Elements"

[This chapter has been adapted from an entry in Wikipedia.[1]]

Euclid's "Elements" is a mathematical and geometric treatise consisting of 13 books written by the ancient Greek mathematician Euclid in Alexandria c.300 BC. It is a collection of definitions, postulates (axioms), propositions (theorems and constructions), and mathematical proofs of the propositions. The thirteen books cover Euclidean geometry and the ancient Greek version of elementary number theory. The work also includes an algebraic system that has become known as geometric algebra, which is powerful enough to solve many algebraic problems, including the problem of finding the square root of a number. With the exception of Autolycus' "On the Moving Sphere", the Elements is one of the oldest extant Greek mathematical treatises, and it is the oldest extant axiomatic deductive treatment of mathematics. It has proven instrumental in the development of logic and modern science. The name "Elements" comes from the plural of "element". According to Proclus, the term was used to describe a theorem that is all-pervading and helps furnishing proofs of many other theorems. The word "element" is in the Greek language the same as "letter": this suggests that theorems in the "Elements" should be seen as standing in the same relation to geometry as letters to language. Later commentators give a slightly different meaning to the term "element", emphasizing how the propositions have progressed in small steps and continued to build on previous propositions in a well-defined order.

Euclid's "Elements" has been referred to as the most successful and influential textbook ever written. Being first set in type in Venice in 1482, it is one of the very earliest mathematical works to be printed after the invention of the printing press and was estimated by Carl Benjamin Boyer to be second only to the Bible in the number of editions published (the number reaching well over one thousand). For centuries, when the *quadrivium* was included in the curriculum of all university students, knowledge of at least part of Euclid's

[1] `http://en.wikipedia.org/wiki/Euclid's_Elements`

"Elements" was required of all students. Not until the 20th century, by which time its content was universally taught through other school textbooks, did it cease to be considered something all educated people had read.

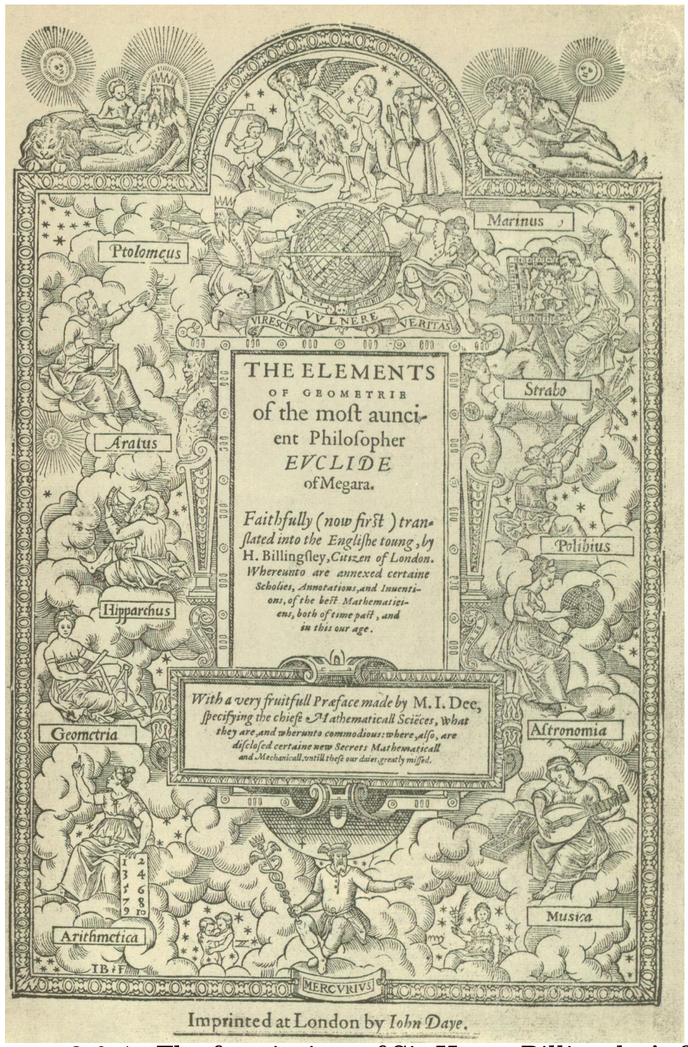

FIGURE 2.0.1. The frontispiece of Sir Henry Billingsley's first English version of Euclid's Elements, 1570.

2.1. History

2.1.1. Basis in earlier work. Scholars believe that the Elements is largely a collection of theorems proved by other mathematicians supplemented by some original work. Proclus, a Greek mathematician who lived several centuries after Euclid, wrote in his commentary: "Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefutable demonstration the things which were only somewhat loosely proved by his predecessors". Pythagoras was probably the source of most of books I and II, Hippocrates of Chios of book III, and Eudoxus book V, while books IV, VI, XI, and XII probably came from other Pythagorean or Athenian mathematicians. Euclid often replaced fallacious proofs with his own more rigorous versions. The use of definitions, postulates, and axioms dated back to Plato. The "Elements" may have been based on an earlier textbook by Hippocrates of Chios, who also may have originated the use of letters to refer to figures.

2.1.2. Transmission of the text. In the fourth century AD, Theon of Alexandria extended an edition of Euclid which was so widely used that it became the only surviving source until François Peyrard's 1808 discovery at the Vatican of a manuscript not derived from Theon's. This manuscript, the Heiberg manuscript, is from a Byzantine workshop c. 900 and is the basis of modern editions. Papyrus Oxyrhynchus 29 is a tiny fragment of an even older manuscript, but only contains the statement of one proposition.

Although known to Cicero, there is no extant record of the text having been translated into Latin prior to Boethius in the fifth or sixth century. The Arabs received the Elements from the Byzantines in approximately 760; this version, by a pupil of Euclid called Proclo, was translated into Arabic under Harun al Rashid c.800. The Byzantine scholar Arethas commissioned the copying of one of the extant Greek manuscripts of Euclid in the late ninth century. Although known in Byzantium, the "Elements" was lost to Western Europe until c. 1120, when the English monk Adelard of Bath translated it into Latin from an Arabic translation.

The first printed edition appeared in 1482 (based on Campanus of Novara's 1260 edition), and since then it has been translated into many languages and published in about a thousand different editions. Theon's Greek edition was recovered in 1533. In 1570, John Dee provided a widely respected "Mathematical Preface", along with copious notes and supplementary material, to the first English edition by Henry Billingsley.

Copies of the Greek text still exist, some of which can be found in the Vatican Library and the Bodleian Library in Oxford. The manuscripts available are of variable quality and are invariably incomplete. By careful analysis of the translations and originals, hypotheses have been made about the contents of the original text (copies of which are no longer available).

Ancient texts which refer to the “Elements” itself and to other mathematical theories that were current at the time it was written are also important in this process. Such analyses are conducted by J.L. Heiberg and Sir Thomas Little Heath in their editions of the text.

Also of importance are the scholia, or annotations, to the text. These additions which often distinguished themselves from the main text (depending on the manuscript) gradually accumulated over time as opinions varied upon what was worthy of explanation or further study.

2.2. Influence

The “Elements” is still considered a masterpiece in the application of logic to mathematics. In historical context, it has proven enormously influential in many areas of science. Scientists Nicolaus Copernicus, Johannes Kepler, Galileo Galilei, and Sir Isaac Newton were all influenced by the Elements, and applied their knowledge of it to their work. Mathematicians and philosophers, such as Bertrand Russell, Alfred North Whitehead, and Baruch Spinoza, have attempted to create their own foundational “Elements” for their respective disciplines by adopting the axiomatized deductive structures that Euclid’s work introduced.

The austere beauty of Euclidean geometry has been seen by many in western culture as a glimpse of an otherworldly system of perfection and certainty. Abraham Lincoln kept a copy of Euclid in his saddlebag, and studied it late at night by lamplight; he related that he said to himself, “You never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father’s house, and stayed there till I could give any proposition in the six books of Euclid at sight.” Edna St. Vincent Millay wrote in her sonnet Euclid Alone Has Looked on Beauty Bare, “O blinding hour, O holy, terrible day, When first the shaft into his vision shone Of light anatomized!” Einstein recalled a copy of the “Elements” and a magnetic compass as two gifts that had a great influence on him as a boy, referring to the Euclid as the "holy little geometry book".

The success of the “Elements” is due primarily to its logical presentation of most of the mathematical knowledge available to Euclid. Much of the material is not original to him, although many of the proofs are his. However, Euclid’s systematic development of his subject, from a small set of axioms to deep results, and the consistency of his approach throughout the Elements, encouraged its use as a textbook for about 2,000 years. The “Elements” still influences modern geometry books. Further, its logical axiomatic approach and rigorous proofs remain the cornerstone of mathematics.

FIGURE 2.2.1. The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid’s “Elements” in 1607.

2.3. Outline of Elements

Books 1 through 4 deal with plane geometry.

Book 2 is commonly called the "book of geometric algebra" because most of the propositions can be seen as geometric interpretations of algebraic identities, such as $a(b + c + ...) = ab + ac + ...$ or $(2a + b)^2 + b^2 = 2(a^2 + (a + b)^2)$. It also contains a method of finding the square root of a given number.

Book 3 deals with circles and their properties: inscribed angles, tangents, the power of a point, Thales' theorem.

Book 4 constructs the incircle and circumcircle of a triangle, and constructs regular polygons with 4, 5, 6, and 15 sides.

Books 5 through 10 introduce ratios and proportions.

Book 5 is a treatise on proportions of magnitudes. Proposition 25 has as a special case the inequality of arithmetic and geometric means.

Book 6 applies proportions to geometry: Similar figures.

Book 7 deals strictly with elementary number theory: divisibility, prime numbers, Euclid's algorithm for finding the greatest common divisor, least common multiple. Propositions 30 and 32 together are essentially equivalent to the fundamental theorem of arithmetic stating that every positive integer can be written as a product of primes in an essentially unique way, though Euclid would have had trouble stating it in this modern form as he did not use the product of more than 3 numbers.

Book 8 deals with proportions in number theory and geometric sequences.

Book 9 applies the results of the preceding two books and gives the infinitude of prime numbers (proposition 20), the sum of a geometric series (proposition 35), and the construction of even perfect numbers (proposition 36).

Book 10 attempts to classify incommensurable (in modern language, irrational) magnitudes by using the method of exhaustion, a precursor to integration.

Books 11 through to 13 deal with spatial geometry: Book 11 generalizes the results of Books 1–6 to space: perpendicularity, parallelism, volumes of parallelepipeds.

Book 12 studies volumes of cones, pyramids, and cylinders in detail, and shows for example that the volume of a cone is a third of the volume of the corresponding cylinder. It concludes by showing the volume of a sphere is proportional to the cube of its radius by approximating it by a union of many pyramids.

Book 13 constructs the five regular Platonic solids inscribed in a sphere, calculates the ratio of their edges to the radius of the sphere, and proves that there are no further regular solids.

2.4. Euclid's method and style of presentation

Euclid's axiomatic approach and constructive methods were widely influential.

As was common in ancient mathematical texts, when a proposition needed proof in several different cases, Euclid often proved only one of them (often the most difficult), leaving the others to the reader. Later editors such as Theon often interpolated their own proofs of these cases.

Euclid's presentation was limited by the mathematical ideas and notations in common currency in his era, and this causes the treatment to seem awkward to the modern reader in some places. For example, there was no notion of an angle greater than two right angles, the number 1 was sometimes treated separately from other positive integers, and as multiplication was treated geometrically; in fact, he did not use the product of more than three different numbers. The geometrical treatment of number theory may have been because the alternative would have been the extremely awkward Alexandrian system of numerals.

The presentation of each result is given in a stylized form, which, although not invented by Euclid, is recognized as typically classical. It has six different parts: first is the statement of the proposition in general terms (also called the enunciation). Then the setting-out, which gives the figure and denotes particular geometrical objects by letters. Next comes the definition or specification which restates the enunciation in terms of the particular figure. Then the construction or machinery follows. It is here that the original figure is extended to forward the proof. The proof itself follows. Finally, the conclusion connects the proof to the enunciation by stating the specific conclusions constructed in the proof in the general terms of the enunciation.

No indication is given of the method of reasoning that led to the result, although the data does provide instruction about how to approach the types of problems encountered in the first four books of the Elements. Some scholars have tried to find fault in Euclid's use of figures in his proofs, accusing him of writing proofs that depended on the specific figures constructed rather than the general underlying logic (especially concerning Proposition II of Book I). However, Euclid's original proof of this proposition is general, valid, and does not depend on the figure used as an example to illustrate one given configuration.

2.4.1. Criticism. While Euclid's list of axioms in the "Elements" is not exhaustive, it represents the most important principles. His proofs often invoke axiomatic notions which were not originally presented in his list of axioms.

Later editors have interpolated Euclid's implicit axiomatic assumptions in the list of formal axioms.

For example, in the first construction of Book 1, Euclid uses a premise that was neither postulated nor proved: that two circles with centers at the distance of their radius will intersect in two points. Later, in the fourth construction, he uses superposition (moving the triangles on top of each other) to prove that if two sides and their angles are equal then they are congruent. During these considerations, he uses some properties of superposition, but these properties are not constructs explicitly in the treatise. If superposition is to be considered a valid method of geometric proof, all of geometry would be full of such proofs. For example, propositions 1.1 – 1.3 can be proved trivially by using superposition.

Mathematician and historian W. W. Rouse Ball puts these criticisms in perspective, remarking that "the fact that for two thousand years ["The Elements"] was the usual text-book on the subject raises a strong presumption that it is not unsuitable for that purpose."

CHAPTER 3

Open Textbooks

[This chapter has been adapted from an entry in Wikipedia.[1]]

An open textbook is a textbook licensed under an open copyright license and made available online to be freely used by students, teachers and members of the public. Many open textbooks are distributed in other printed, e-book, or audio formats that may be downloaded or purchased at little or no cost.

Part of the broader open educational resources movement, open textbooks increasingly are seen as a solution to challenges with traditionally published textbooks, such as access and affordability concerns. Open textbooks were identified in the New Media Consortium's *2010 Horizon Report* as a component of the rapidly progressing adoption of open content in higher education.

3.1. Usage Rights

The defining difference between open textbooks and traditional textbooks is that the copyright permissions on open textbooks allow the public to freely use, adapt, and distribute the material. Open textbooks either reside in the public domain or are released under an open license that grants usage rights to the public so long as the author is attributed.

The copyright permissions on open textbooks extend to all members of the public and cannot be rescinded. These permissions include the right to do the following:

- use the textbook freely
- create and distribute copies of the textbook
- adapt the textbook by revising it or combining it with other materials

Some open licenses limit these rights to non-commercial use or require that adapted versions be licensed the same as the original.

3.2. Open Licenses

Some examples of open licenses are:

- Creative Commons Attribution (CC-BY)

[1] https://en.wikipedia.org/wiki/Open_textbook

- Creative Commons Attribution Share-Alike (CC-BY-SA)
- Creative Commons Attribution Non-Commercial Share-Alike (CC-BY-NC-SA)
- GNU Free Documentation License

Waivers of copyright that place materials in the public domain include:

- Creative Commons Public Domain Certification

3.3. Affordability

Open textbooks increasingly are seen as an affordable alternative to traditional textbooks in both K-12 and higher education. In both cases, open textbooks offer both dramatic up-front savings and the potential to drive down traditional textbook prices through competition.

3.3.1. Higher Education. Overall, open textbooks have been found by the Student PIRGs to offer 80% or more savings to higher education students over traditional textbook publishers. Research commissioned by the Florida State Legislature found similarly high savings and the state has since implemented a system to facilitate adoption of open textbooks.

In the Florida legislative report, the governmental panel found after substantial consultation with educators, students, and administrators that "there are compelling academic reasons to use open access textbooks such as: improved quality, flexibility and access to resources, interactive and active learning experiences, currency of textbook information, broader professional collaboration, and the use of teaching and learning technology to enhance educational experiences." (OATTF, p. i) Similar state-backed initiatives are underway in Washington, Ohio, California, and Texas. In Canada, the province of British Columbia became the first jurisdiction to have a similar open textbook program.

3.3.2. K–12 Education. Research at Brigham Young University has produced a web-based cost comparison calculator for traditional and open K-12 textbooks. To use the calculator the inputs commercial textbook cost, planned replacement frequency, and number of annual textbook user count are required. A section is provided to input time requirements for adaptation to local needs, annual updating hours, labor rate, and an approximation of pages. The summary section applies an industry standard cost for print-on-demand of the adapted open textbook to provide a cost per student per year for both textbook options. A summed cost differential over the planned period of use is also calculated.

3.4. Milestones

In November 2010, Dr. Anthony Brandt was awarded an "Access to Artistic Excellence" grant from the National Endowment for the Arts for his innovative music appreciation course in Connexions. "Sound Reasoning ... takes a new approach [to teaching music appreciation]: It presents style-transcendent principles, illustrated by side-by-side examples from both traditional and contemporary music. The goal is to empower listeners to be able to listen attentively and think intelligently about any kind of music, no matter its style. Everything is listening based; no ability to read music is required." The module being completed with grant funds is entitled "Hearing Harmony". Dr. Brandt cites choosing the Connexions open content publishing platform because "it was an opportunity to present an innovative approach in an innovative format, with the musical examples interpolated directly into the text."

In December 2010, open textbook publisher Flat World Knowledge was recognized by the American Library Association's Business Reference and Services Section (ALA BRASS) by being named to the association's list of "Outstanding Business Reference Sources: The 2010 Selection of Recent Titles." The categories of business and economics open textbooks from Flat World Knowledge's catalog were selected for this award and referenced as "an innovative new vehicle for affordable (or free) online access to premier instructional resources in business and economics." Specific criteria used by the American Library Association BRASS when evaluating titles for selection were:

> A resource compiled specifically to supply information on a certain subject or group of subjects in a form that will facilitate its ease of use. The works are examined for authority and reputation of the publisher, author, or editor; accuracy; appropriate bibliography; organization, comprehensiveness, and value of the content; currency and unique addition to the field; ease of use for intended purpose; quality and accuracy of indexing; and quality and usefulness of graphics and illustrations. Each year more electronic reference titles are published, and additional criteria by which these resources are evaluated include search features, stability of content, graphic design quality, and accuracy of links. Works selected are intended to be suitable for medium to large academic and public libraries.

Because authors do not make money form the sale of open textbooks, several organizations have tried to use prizes or grants as financial incentives for writing open textbooks or releasing existing textbooks under open licenses. Connexions announced a series of two grants in early 2011 that will allow them to produce a total of 20 open textbooks. The first five titles will be produced over an 18 month time frame for Anatomy & Physiology, Sociology, Biology, Biology for non-majors, and Physics. The second phase will produce an additional 15 titles with subjects that have yet to be determined. It is noted the most expensive part of producing an open textbook is image rights clearing. As images are cleared for this project, they will be available for reuse in even more titles. In addition, the Saylor Foundation sponsors an ongoing "Open Textbook Challenge", offering a $20,000 reward for newly-written open textbooks or existing textbooks released under a CC-BY license.

The Text and Academic Author's Association awarded a 2011 Textbook Excellence Award ("Texty") to the first open textbook to ever win such recognition this year. A maximum of eight academic titles can earn this award each year. The title "Organizational Behavior" by Talya Bauer and Berrin Erdogan earned one of seven 2011 Textbook Excellence Awards granted. Bauer & Erdogan's "Organizational Behavior" open textbook is published by Flat World Knowledge.

3.5. Instruction

Open textbooks are flexible in ways that traditional textbooks are not, which gives instructors more freedom to use them in the way that best meets their instructional needs.

One common frustration with traditional textbooks is the frequency of new editions, which force the instructor to modify the curriculum to the new book. Any open textbook can be used indefinitely, so instructors need only change editions when they think it is necessary.

Many open textbooks are licensed to allow modification. This means that instructors can add, remove or alter the content to better fit a course's needs. Furthermore, the cost of textbooks can in some cases contribute to the quality of instruction when students are not able to purchase required materials. A Florida governmental panel found after substantial consultation with educators, students, and administrators that "there are compelling academic reasons to use open access textbooks such as: improved quality, flexibility and access to resources, interactive and active learning experiences, currency of textbook information, broader professional collaboration, and the use of teaching and learning technology to enhance educational experiences." (OATTF, p. i)

3.6. Authorship

Author compensation for open textbooks works differently than traditional textbook publishing. By definition, the author of an open textbook grants the public the right to use the textbook for free, so charging for access is no longer possible. However, numerous models for supporting authors are developing. For example, a start-up open textbook publisher called Flat World Knowledge pays its authors royalties on the sale of print copies and study aids. Other proposed models include grants, institutional support and advertising.

3.7. Projects

A number of projects seek to develop, support and promote open textbooks. Two very notable advocates and supporters of open textbook and related open education projects include the William and Flora Hewlett Foundation and the Bill and Melinda Gates Foundation.

CHAPTER 4

Recommended Reading

1. "Geometry: Seeing, Doing, Understanding" 3rd edition, by Harold R. Jacobs (ISBN: 978-0716743613). I recommend this title for beginning geometry students as a primary textbook along with "Euclid's 'Elements' Redux" as a secondary textbook. (An Enhanced Teacher's Guide and an Improved Test Bank are also available, although both appear to be out of print.) The textbook's ISBN: 978-0716743613

2. "Book of Proof" 2nd edition, by Richard Hammack. This open textbook is an introduction to the standard methods of proving mathematical theorems. It can be considered a companion volume to any edition of Euclid, especially for those who are learning how to read and write mathematical proofs for the first time. It has been approved by the American Institute of Mathematics' Open Textbook Initiative and has a number of good reviews at the Mathematical Association of America Math DL and on Amazon. Visit the website at:

`http://www.people.vcu.edu/~rhammack/BookOfProof/index.html`

3. Math Open Reference[1], especially the topic of Triangle Centers[2].

4. Khan Academy[3]

5. "The Thirteen Books of Euclid's Elements", translation and commentaries by Sir Thomas Heath in three volumes. Published by Dover Publications, Vol. 1: ISBN 978-0486600888, Vol. 2: ISBN 978-0486600895, Vol. 3: ISBN 978-0486600901.

6. "Euclid's Elements – All thirteen books in one volume". Based on Heath's translation, Green Lion Press, ISBN 978-1888009194.

[1] `http://www.mathopenref.com/`
[2] `http://www.mathopenref.com/trianglecenters.html`
[3] `https://www.khanacademy.org/`

Part 2

The “Elements”

In this textbook, students are expected to construct figures as they are given, step-by-step. This is an essential component to the learning process that cannot be avoided. In fact, this is the impetus behind the historical quote, “There is no royal road to geometry.” That is, no one learns mathematics “for free”.

The propositions of Euclid will be referred to as (for example) either Proposition 3.32 or [3.32], with chapter and proposition number separated by a period. Axioms, Definitions, etc., will also be referred to in this way: for example, Definition 12 in chapter 1 will be denoted as [Def. 1.12]. Exercises to problems will be denoted as (for example) [3.5, #1] for exercise 1 of Proposition 3.5.

A note on exercises: generally, an exercise is expected to be solved using the propositions, corollaries, and exercises that preceded it. For example, exercise [1.32, #3] should first be attempted using propositions [1.1]-[1.32] as well as all previous exercises. Should this prove too difficult or too frustrating for the student, then he/she should consider whether propositions [1.33] or later (and their exercises) might help solve the exercise. It is also permissible to use trigonometry, linear algebra, or other contemporary mathematical techniques on challenging problems.

CHAPTER 1

Angles, Parallel Lines, Parallelograms

The following symbols will be used to denote certain standard geometric shapes or relationships:

- Circles will be denoted by: $\circ$
- Triangles by: $\triangle$
- Parallelograms by: $\boxdot$
- Parallel lines by: $\|$
- Perpendicular lines by: $\perp$

In addition to these, we shall employ the usual symbols of algebra, $+$, $-$, $=$, $<$, $>$, $\neq$, $\not<$, $\not>$, as well as two additional symbols:

- Composition: $\oplus$ For example, suppose we have the segments AB and BC which intersect at the point B. The statement $AB+BC$ refers to the sum of their lengths, but $AB \oplus BC$ refers to their composition as one object. See Fig. 1.0.1.

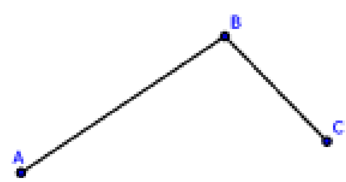

FIGURE 1.0.1. Composition: the geometrical object $AB \oplus BC$ is a single object composed of two segments, AB and BC.

The composition of angles, however, can be written using either $+$ or $\oplus$, and in this textbook their composition will be denoted with $+$.

- Congruence: $\cong$ Two figures or objects are congruent if they have the same shape and size, or if one has the same shape and size as the mirror image of the other. This means that an object can be repositioned and reflected (but not re-sized) so as to coincide precisely with the other object.[1]

[1] http://en.wikipedia.org/wiki/Congruence_(geometry)

- Similar: $\sim$ Two figures or objects are similar if they have the same shape but not necessarily the same size. If two similar objects have the same size, they are also congruent.

1.1. Definitions

The Point. 1. A *point* is a zero dimensional object.[2] A geometrical object which has three dimensions (length, height, and width) is a solid. A geometrical object which has two dimensions (length and height) is a surface, and a geometrical object which has one dimension only is a line. Since a point has none of these, it has zero dimensions.

The Line. 2. A *line* is a one dimensional object: it has only length. If it had any height, no matter how small, it would be space of two dimensions. If it had any width, it would be space of three dimensions. Hence, a line has neither height nor width.

(This definition conforms to Euclid's original definition in which a line need not be straight. However, in all modern geometry texts, it is understood that a "line" has no curves. See also [Def 1.4].)

3. The intersections of lines are points.

4. A line without a curve between its endpoints is called a *straight line*. It is understood throughout this textbook that a *line* refers exclusively to a *straight line*. A curved line (such as the circumference of a circle) will never be referred to merely as a line in order to avoid confusion. Lines have no endpoints since they are infinite in length.

A *line segment* or more simply a *segment* is like a line except that it is finite in length and has two *endpoints* which occur at its extremities.

A *ray* is like a line in that it is infinite in length; however, it has only one endpoint. See Fig. 1.1.1.

[2]Warren Buck, Chi Woo, Giangiacomo Gerla, J. Pahikkala. "point" (version 13). PlanetMath.org. Freely available at `http://planetmath.org/point`

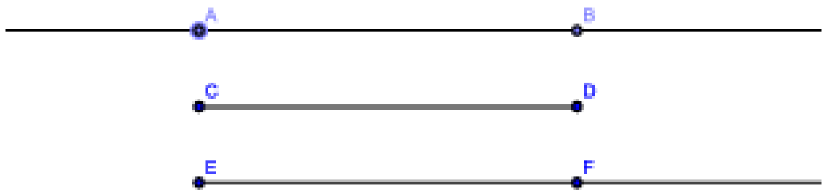

FIGURE 1.1.1. [Def. 1.2, 1.3, 1.4] AB is a line, CD is a segment, and EF is a ray

The Plane. 5. A *surface* has two dimensions, length and height. It has no width; if it had, however small, it would be space of three dimensions.

6. A surface is called a *plane* whenever two arbitrary points on the surface can be joined by a right angle.

7. Any combination of points, of lines, or of points and lines on a plane is called a *plane figure*. A plane figure that is bounded by a finite number of straight line segments closing in a loop to form a closed chain or circuit is called a *polygon*[3].

8. Points which lie on the same straight line, ray, or segment are called *collinear points*.

The Angle. 9. The angle made by of two straight lines, segments, or rays extending outward from a common point but in different directions is called a *rectilinear angle* or simply an *angle.*

10. The common point of intersection between straight lines, rays, or segments is called the *vertex of the angle*.

11. A particular angle in a figure will be denoted by the symbol $\angle$ and three letters, such as BAC, of which the middle letter, A, is at the vertex. Hence, an angle may be referred to either as $\angle BAC$ or $\angle CAB$. Occasionally, this notation will be shortened to "the angle at point A" instead of naming the angle as above. See Fig. 1.1.2.

[3] http://en.wikipedia.org/wiki/Polygon

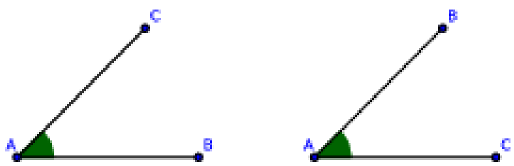

FIGURE 1.1.2. [Def 1.11] Notice that both angles could be referred to as $\angle BAC$, $\angle CAB$, or the angle at point A. Also note that A is a vertex.

12. The angle formed by composing two or more angles is called their sum. Thus in Fig. 1.1.3, we have that $\angle ABC \oplus \angle PQR = \angle ABR$ where the segment QP is applied to the segment BC such that the vertex Q falls on the vertex B and the side QR falls on the opposite side of BC from BA. We generally choose to write $\angle ABC + \angle PQR = \angle ABR$ to express the same concept.

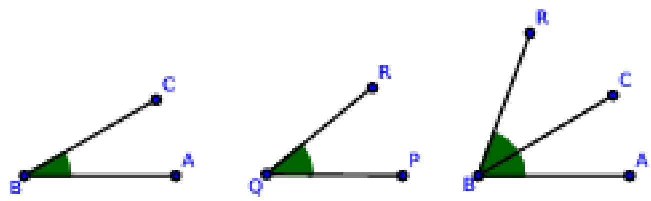

FIGURE 1.1.3. [Def. 1.12]

13. When two segments BA, AD are composed such that $BA \oplus AD = BD$ where BD is another segment, the angles $\angle BAC$ and $\angle CAD$ are called *supplements* of each other (see Fig. 1.1.4). This definition holds when we replace segments by straight lines or rays, *mutatis mutandis*[4].

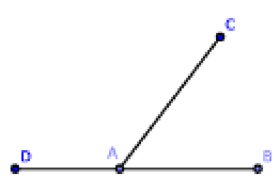

FIGURE 1.1.4. [Def. 1.13]

[4]*Mutatis mutandis* is a Latin phrase meaning "changing [only] those things which need to be changed" or more simply "[only] the necessary changes having been made". Source: http://en.wikipedia.org/wiki/Mutatis_mutandis

14. When one segment (AC) stands on another (DB) such that the adjacent angles on either side of the first segment are equal (that is, $\angle DAC = \angle CAB$), each of the angles is called a *right angle*, and the segment which stands on the other is described as *perpendicular* to the other (or it is called *the perpendicular* to the other). See Fig. 1.1.5. We may also write that AC is perpendicular to DB or more simply that $AC \perp DB$. It follows that the supplementary angle of a right angle is another right angle.

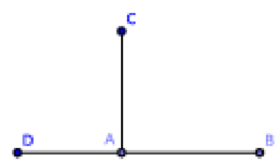

FIGURE 1.1.5. [Def. 1.14]

Multiple perpendicular lines on a many-sided object may be referred to as the object's *perpendiculars*.

The above definition holds for straight lines and rays, *mutatis mutandis*.

A line segment within a triangle from a vertex to an opposite side which is also perpendicular to that side is usually referred to an *altitude* of the triangle, although it could in a general sense be referred to as a perpendicular of the triangle.

15. An *acute angle* is one which is less than a right angle. $\angle CAB$ in Fig. 1.1.4 and $\angle DAB$ is Fig. 1.1.6 are acute angles.

16. An *obtuse angle* is one which is greater than a right angle. $\angle CAD$ in Fig. 1.1.4 is an obtuse angle. The supplement of an acute angle is obtuse, and conversely, the supplement of an obtuse angle is acute.

17. When the sum of two angles is a right angle, each is called the *complement* of the other.

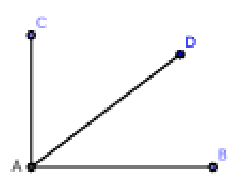

FIGURE 1.1.6. [Def. 1.17] The angle $\angle BAC$ is a right angle. Since $\angle BAC = \angle CAD + \angle DAB$, it follows that the angles $\angle BAD$, $\angle DAC$ are each complements of the other.

Concurrent Lines. 18. Three or more straight lines intersecting the same point are called *concurrent* lines. This definition holds for rays and segments, *mutatis mutandis*.

19. A system of more than three concurrent lines is called a *pencil of lines*. The common point through which the rays pass is called the vertex .

The Triangle. 20. A triangle is a polygon formed by three segments joined at their endpoints. These three segments are called the *sides* of the triangle. One side in particular may be referred to as the *base* of the triangle for explanatory reasons, but there is no fundamental difference between the properties of a base and either of the two remaining sides of a triangle.

21. A triangle whose three sides are unequal in length is called *scalene* (the left-hand example in Fig. 1.1.7). A triangle with two equal sides is called *isosceles* (the middle example in Fig. 1.1.7). When all sides are equal, a triangle is called *equilateral*, (the right-hand example in Fig. 1.1.7). When all angles are equal, a triangle is called *equiangular.*

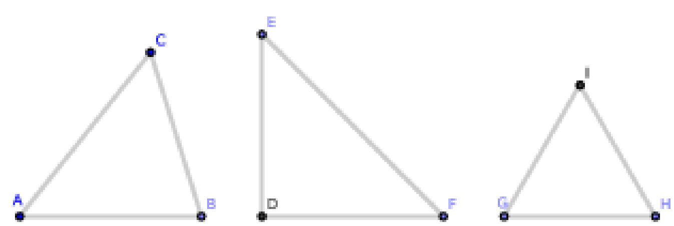

FIGURE 1.1.7. [Def 3.21] The three types of triangles: *scalene, isosceles, equilateral.*

22. A *right triangle* is a triangle in which one of its angles is a right angle, such as the middle example in Fig. 1.1.7. The side which stands opposite the right angle is called the *hypotenuse* of the triangle. (In the middle example in Fig. 1.1.7, $\angle EDF$ is a right angle, so side EF is the hypotenuse of the triangle.)

23. An *obtuse* triangle is a triangle such that one of its angles obtuse (such as $\triangle CAB$ in Fig. 1.1.8).

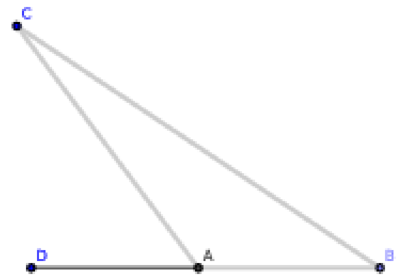

FIGURE 1.1.8. [Def. 1.23]

24. An *acute triangle* is a triangle such that each of its angles are acute, such as the left- and right-hand examples in Fig. 1.1.7.

25. An *exterior angle* of a triangle is one which is formed by extending the side of a triangle. For example, the triangle in Fig. 1.1.8 has had side BA extended to the segment AD which creates the exterior angle $\angle DAC$.

Every triangle has six exterior angles. Also, each exterior angle is the supplement of the adjacent interior angle. In Fig. 1.1.8, the exterior angle $\angle DAC$ is the supplement of the adjacent interior angle $\angle CAB$.

The Polygon. 26. A *rectilinear figure* bounded by three or more segments is referred to as a *polygon*. For example, the object in Fig. 1.0.1 is a plane figure but not a polygon. The triangles in Fig. 1.1.7 are both plane figures and polygons.

27. A *polygon* is said to be convex when it has no re-entrant angle (that is, it does not have an interior angle greater than $180°$).

28. A polygon of four sides is called a *quadrilateral*.

29. A quadrilateral whose four sides are equal in length is called a *lozenge*. A lozenge is also a form of rhombus[5] and therefore also a parallelogram.

30. A rhombus which has a right angle is called a *square*.

31. A polygon which has five sides is called a *pentagon*; one which has six sides, a *hexagon*, etc.[6]

[5] https://en.wikipedia.org/wiki/Rhombus

[6] See also https://en.wikipedia.org/wiki/Polygon

The Circle. 32. A *circle* is a plane figure formed by a curved line called the *circumference* such that all segments constructed from a certain point within the figure to the circumference are equal in length. That point is called the *center* of the circle.

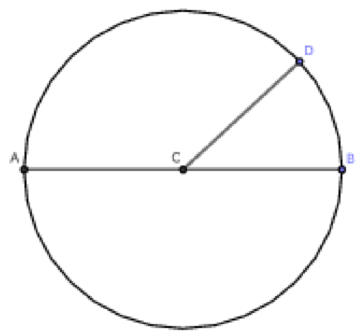

FIGURE 1.1.9. [Def. 1.32] $\circ BDA$ constructed with center C and radius CD. Notice that $CA = CB = CD$. Also notice that AB is a diameter.

33. A *radius* of a circle is any segment constructed from the center to the circumference, such as CA, CB, CD in Fig. 1.1.9. Notice that $CA = CB = CD$.

34. A *diameter* of a circle is a segment constructed through the center and terminated in both directions by the circumference, such as AB in Fig. 1.1.9.

From the definition of a circle, it follows that the path of a movable point in a plane which remains at a constant distance from a fixed point is a circle. Also, any point P in the plane is either inside, outside, or on the circumference of a circle depending on whether its distance from the center is less than, greater than, or equal to the radius.

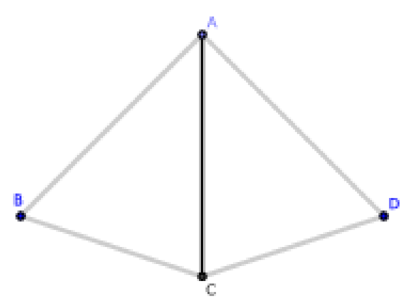

FIGURE 1.1.10. [Def. 1.35]

Other. 35. A segment, line, or ray in any figure which divides the area of a geometric object into two equal halves is called an *Axis of Symmetry* of the figure (such as AC in the polygon $ABCD$, Fig. 1.1.10).

36. A segment constructed from any angle of a triangle to the midpoint of the opposite side is called a *median of the triangle*. Each triangle has three medians which are concurrent. The point of intersection of the three medians is called the centroid of the triangle.

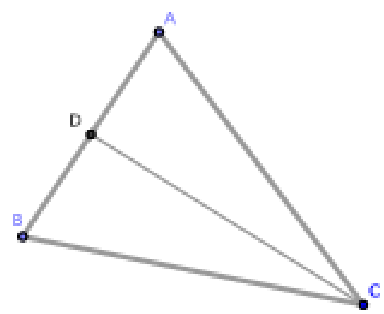

FIGURE 1.1.11. [Def. 1.36] Segment CD is a median of $\triangle ABC$.

37. A locus (plural: loci) is a set of points whose location satisfies or is determined by one or more specified conditions, i.e., 1) every point satisfies a given condition, and 2) every point satisfying it is in that particular locus.[7] For example, a circle is the locus of a point whose distance from the center is equal to its radius.

1.2. Postulates

We assume the following:

(1) A straight line, ray, or segment may be constructed from any one point to any other point. Lines, rays, and segments may be subdivided by points into segments or subsegments which are finite in length.
(2) A segment may be extended from any length to a longer segment, a ray, or a straight line.
(3) A circle may be constructed from any point (which we denote as its center) and from any finite length measured from the center (which we denote as its radius).

[7] http://en.wikipedia.org/wiki/Locus_(mathematics)

Note: if we have constructed two points A and B on a sheet of paper, and if we construct a segment from A to B, this segment will have some irregularities due to the spread of ink or slight flaws in the paper, both of which introduce some height and width. Hence, it will not be a true geometrical segment no matter how nearly it may appear to be one. This is the reason that Euclid postulates the construction of segments, rays, and straight lines from one point to another (where our choice of paper, application, etc., is irrelevant). For if a segment could be accurately constructed, there would be no need for Euclid to ask us to take such an action for granted. Similar observations apply to the other postulates. It is also worth nothing that Euclid never takes for granted the accomplishment of any task for which a geometrical construction, founded on other problems or on the foregoing postulates, can be provided.

1.3. Axioms

Axioms 1-7 and 9 hold for every kind and variety of magnitude. Axioms 8 and 10-12 are strictly geometrical. Note that all Euclidean magnitudes are positive.

(1) If we consider three magnitudes such that the first magnitude is equal to the second and the second magnitude is equal to the third, we infer that the first magnitude equals the third.
 (a) If $A = B$, and $B = C$, then $A = C$.
(2) If equals are added to equals, then their sums are equal.
 (a) If $A = B$ and C is added to both A and B, then $A + C = B + C$.
(3) If equals are taken from equals, then the remainders are equal.
 (a) If $A = B$ and C is subtracted from both A and B, then $A - C = B - C$.
(4) If equals are added to unequals, then the sums are unequal.
 (a) If $A > B$ and C is added to both A and B, then $A + C > B + C$.
 (b) If $A < B$ and C is added to both A and B, then $A + C < B + C$.
(5) If equals are taken from unequals, then the remainders are unequal.
 (a) If $A > B$ and C is subtracted from both A and B, then $A - C > B - C$.
 (b) If $A < B$ and C is subtracted from both A and B, then $A - C < B - C$.
(6) The doubles of equal magnitudes are equal.
 (a) If $A = B$, then $2A = 2B$.
(7) The halves of equal magnitudes are equal.
 (a) If $A = B$, then $A/2 = B/2$.

(8) Magnitudes which can be made to coincide are equal.
   (a) The placing of one geometrical object on another, such as a line on a line, a triangle on a triangle, or a circle on a circle, etc., is called *superposition*. The superposition employed in geometry is only mental; that is, we conceive of one object being placed on the other. And then, if we can prove that the objects coincide, we infer by the present axiom that they are equal in all respects, including magnitude. Superposition involves the following principle which, without being explicitly stated, Euclid makes frequent use: "Any figure may be transferred from one position to another without change in size or form."
(9) The whole is equal to the sum of all its parts. This is sometimes stated as: the whole is greater than the sum of its parts.
(10) Two straight lines cannot enclose a space.
   (a) This is equivalent to the statement, "If two straight lines have two points common to both, then they coincide in direction." Alternatively, we say that they form a single line because they coincide at every point.
   (b) The above holds for segments and rays, *mutatis mutandis*.
(11) All right angles are equal to each other.
   (a) A proof: Let there be two straight lines AB, CD, and two perpendiculars to them, namely, EF, GH. Then if AB, CD are made to coincide by superposition, so that the point E will coincide with G, then since a right angle is equal to its supplement, the line EF must coincide with GH. Hence $\angle AEF = \angle CGH$.
(12) If two straight lines (AB, CD) intersect a third straight line (AC) such that the sum of the two interior angles ($\angle BAC$, $\angle ACD$) on the same side equals less than two right angles, then if these lines will meet at some finite distance. (This axiom is the converse of [1.17].) See Fig. 1.3.1.

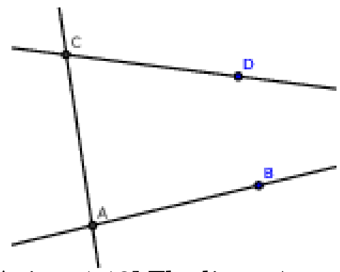

FIGURE 1.3.1. [Axiom 1.12] The lines AB and CD must eventually meet (intersect) at some finite distance.

The above holds for rays and segments, *mutatis mutandis*.

1.4. Explanation of Terms

Axioms: "Elements of human reason" are certain general propositions, the truths of which are self-evident, and which are so fundamental that they cannot be inferred from any propositions which are more elementary. In other words, they are incapable of demonstration. "That two sides of a triangle are greater than the third" is, perhaps, self-evident; but it is not an axiom since it can be inferred by demonstration from other propositions. However, we can give no proof of the proposition that "two objects which are equal in length to a third object are also equal in length to each other". Since that statement is self-evident, it is considered an axiom.

Propositions which are not axioms are properties of figures obtained by processes of reasoning. They may be divided into theorems and problems.

A *theorem* is the formal statement of a property that may be demonstrated from known propositions. These propositions may themselves be theorems or axioms. A theorem consists of two parts: the *hypothesis*, or that which is assumed, and the *conclusion*, or that which is asserted to follow from the argument. We present four examples:

THEOREM. *(1) If X is Y, then Z is W.*

we have that the hypothesis is that X is Y, and the conclusion is that Z is W.

Converse Theorems: Two theorems are said to be converses when the hypothesis of either is the conclusion of the other. Thus the converse of the theorem (1) is:

THEOREM. *(2) If Z is W, then X is Y.*

From two theorems (1) and (2), we may infer two others called their *contrapositives*. The contrapositive of (1) is:

THEOREM. *(3) If Z is not W, then X is not Y.*

The contrapositive of (2) is:

THEOREM. *(4) If X is not Y, then Z is not W.*

Theorem (4) is also called the inverse[8] of (1), and (3) is the inverse of (2).

A *problem* is a proposition in which something is proposed to be done, such as a line or a figure to be constructed under some given conditions.

The *solution* of a problem is the method of construction which accomplishes the required result.

In the case of a theorem, the *demonstration* is the proof that the conclusion follows from the hypothesis. In the case of a problem, the *demonstration* is the construction which creates the proposed object.

The *statement* or *enunciation* of a problem consists of two parts: the *data*, or that which we assume we have to work with, and that which we must accomplish.

Postulates are the elements of geometrical construction and have the same relation with respect to problems as axioms do to theorems.

A *corollary* is an inference or deduction from a proposition.

A *lemma* is an auxiliary proposition required in the demonstration of a principal proposition.

A *secant line* is a line which cuts (intersects) a system of lines, a circle, or any other geometrical figure.

Congruent figures are those that can be made to coincide by superposition. They agree in shape and size but differ in position. Hence by [Axiom 1.8], it follows that corresponding parts or portions of congruent figures are congruent and that congruent figures are equal in every respect.

The *Rule of Symmetry:* If $X = Y$, it follows that $Y = X$.

[8]"The counterpart of a proposition obtained by exchanging the affirmative for the negative quality of the whole proposition and then negating the predicate: The inverse of "'Every act is predictable' is 'No act is unpredictable.'"
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1.5. Propositions from Book I: 1-26

PROPOSITION 1.1. *CONSTRUCTING AN EQUILATERAL TRIANGLE. Given an arbitrary segment, it is possible to construct an equilateral triangle on that segment.*

PROOF. We wish to construct an equilateral triangle on the segment AB.

With A as the center of a circle and AB as its radius, we construct the circle $\circ BCD$ [Postulate 1.3]. With B as center and BA as radius, we construct the circle $\circ ACE$, cutting $\circ BCD$ at point C. Connect segments CA, CB [Postulate 1.1]. We claim that $\triangle ABC$ is the required equilateral triangle.

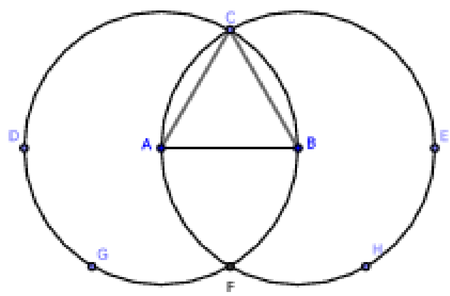

FIGURE 1.5.1. [1.1]

Because A is the center of the circle $\circ BCD$, $AC = AB$ [Def. 1.33]. Since B is the center of the circle $\circ ACE$, $BA = BC$. Since $AB = BA$ (i.e., denoting a segment by its endpoints reading from left to right or from right to left does not affect the segment's length), by [Axiom 1.1], we have that $AC = AB = BA = BC$, or simply $AC = AB = BC$.

Hence, $\triangle ABC$ is an equilateral triangle [Def. 1.21]. Since $\triangle ABC$ is constructed on the given segment AB, the proof follows. □

Examination questions.

1. What is assumed in this proposition?
2. What is that we were to have accomplished?
3. What is a finite straight line?
4. What is the opposite of finite?
5. What postulates were cited and where were they cited?
6. What axioms were cited and where were they cited?
7. What use is made of the definition of a circle? What is a circle?
8. What is an equilateral triangle?

Exercises.

The following exercises use Fig. 1.5.1 and are to be solved when the student has completed Chapter 1.

1. If the segments AF, BF are joined, prove that the figure $\boxdot ACBF$ is a rhombus.

2. If AB is extended to the circumferences of the circles (at points D and E), prove that the triangles $\triangle CDF$ and $\triangle CEF$ are equilateral.

3. If CA, CB are extended to intersect the circumferences at points G and H, prove that the points G, F, H are collinear and that the triangle $\triangle GCH$ is equilateral.

4. Connect CF and prove that $CF^2 = 3AB^2$.

5. Construct a circle in the space ACB bounded by the segment AB and the partial circumferences of the two circles.

PROPOSITION 1.2. *CONSTRUCTING A STRAIGHT-LINE SEGMENT EQUAL TO AN ARBITRARY STRAIGHT-LINE SEGMENT. Given an arbitrary point and an arbitrary segment, it is possible to construct a segment with one endpoint being the previously given point such that its length is equal to that of the arbitrary segment.*

PROOF. Let A be an arbitrary point on the plane, and let BC be an arbitrary segment. We wish to construct a segment with point A as an endpoint and with length equal to that of BC.

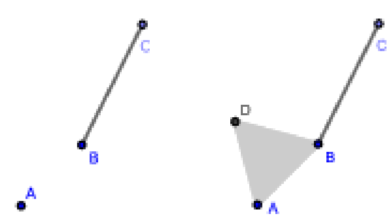

FIGURE 1.5.2. [1.2] partially constructed

On AB, construct the equilateral triangle $\triangle ABD$ [1.1]. With B as the center and BC as the radius, construct the circle $\circ ECH$ [Postulate 1.3]. Extend DB to meet the circle $\circ ECH$ at E [Postulate 1.2]. With D as the center and DE as radius, construct the circle $\circ EFG$ [Postulate 1.3]. Extend DA to meet $\circ EFG$ at F. We claim that $AF = BC$.

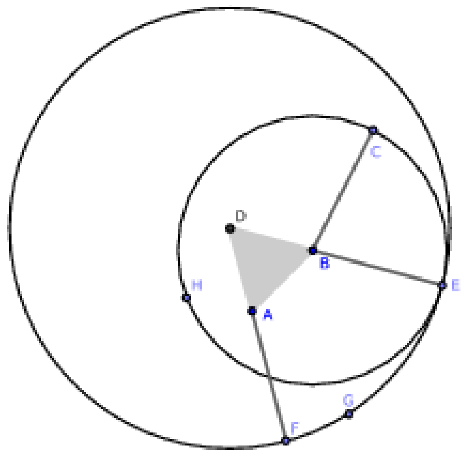

FIGURE 1.5.3. [1.2] fully constructed

Because D is the center of the circle $\circ EFG$, $DF = DE$ [Def. 1.32]. Because $\triangle DAB$ is an equilateral triangle, $DA = DB$ [Def. 1.21]. Removing DA from DF and DB from DE, we have that $AF = BE$ [Axiom 1.3]; that is,

$$\left\{ \begin{array}{lcl} DF - DA & = & DE - DB \\ DF - DA & = & AF \\ DE - DB & = & BE \end{array} \right\} \Longrightarrow AF = BE$$

Again, because B is the center of the circle $\circ ECH$, $BC = BE$. Since $AF = BE$, by [Axiom 1.1] we have that $AF = BC$. Therefore from the given point A, the segment AF has been constructed such that $AF = BC$. □

Exercises.

1. Prove [1.2] when A is a point on BC.

PROPOSITION 1.3. *CUTTING A STRAIGHT-LINE SEGMENT AT A GIVEN SIZE. Given two arbitrary segments which are unequal in length, it is possible to cut the larger segment such that one of its two subsegments is equal in length to the smaller segment.*

PROOF. Let the arbitrary segments CG and AB be constructed such that $AB > CG$. We wish to show that AB may be subdivided into segments AE and EB where $AE = CG$.

From the point A, construct the segment AD such that $AD = CG$ [1.2]. With A as the center and AD as radius, construct the circle $\circ EDF$ [Postulate 1.3] which cuts AB at E. We claim that $AE = CG$.

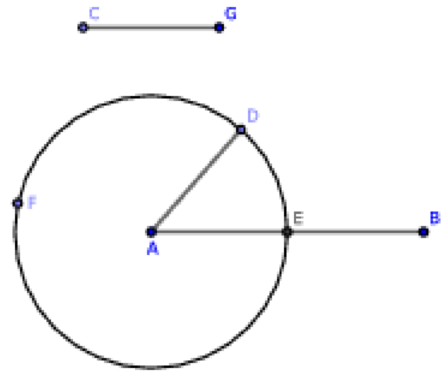

FIGURE 1.5.4. [1.3]

Because A is the center of the circle $\circ EDF$, $AE = AD$ [Def. 1.32]. Also, $AD = CG$ by construction. By [Axiom 1.1], we have that $AE = CG$. □

COROLLARY. *1.4.1. Given an arbitrary segments and a ray, it is possible to cut the ray such that the resulting segment is equal in length to the arbitrary segment.*

Examination questions.

1. What previous problem is employed in the solution of this?
2. What postulate?
3. What axiom is employed in the demonstration?
4. Demonstrate how to extend the lesser of the two given segments until the whole extended segment is equal to the greater segment.

PROPOSITION 1.4. *THE "SIDE-ANGLE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If two pairs of sides of two triangles are equal in length and the corresponding interior angles are equal in measurement, then the triangles are congruent.*

PROOF. Suppose we have triangles $\triangle ABC$ and $\triangle DEF$ such that:

(a) the length of side AB of triangle $\triangle ABC$ is equal in length to side DE of triangle $\triangle DEF$,

(b) the length of side AC of triangle $\triangle ABC$ is equal in length to side DF of triangle $\triangle DEF$,

(c) the measure of the angle $\angle BAC$ is equal in measure to the angle $\angle EDF$.

We wish to show that $\triangle ABC \cong \triangle DEF$.

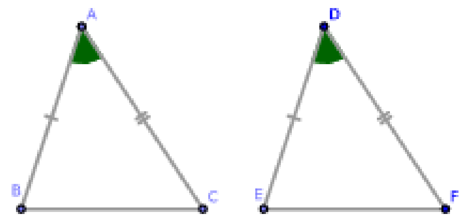

FIGURE 1.5.5. [1.4]

Suppose that $\triangle BAC$ is applied to $\triangle EDF$ such that point A coincides with D, the segment AB coincides with segment DE, and point C stands on the same side of the segment AB as F is relative to DE. Because $AB = DE$, the point B coincides with point E.

Because $\angle BAC = \angle EDF$, AC coincides with DF; and since $AC = DF$ by hypothesis, the point C coincides with point F. This proves that point B coincides with point E. Hence two points of the segment BC coincide with two points of the segment EF. And since two segments cannot enclose a space, BC must coincide with EF. Hence the triangles agree in every respect: $BC = EF$, $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, from which it follows that $\triangle ABC \cong \triangle DEF$. □

Examination questions.

1. How many assumptions do we make in the hypothesis of this proposition? (Ans. 3. Name them.)

2. How many in the conclusion? Name them.

3. What technical term is applied to figures which agree in everything but position?

4. What is meant by superposition?

5. What axiom is made use of in superposition?

6. How many parts in a triangle? (Ans. 6, three sides and three angles.)

7. When it is required to prove that two triangles are congruent, how many parts of one must be given equal to corresponding parts of the other? (Ans. In general, any three except the three angles. This will be established in [1.8] and [1.26], both of which use [1.4].)

8. What property of two segments having two common points is quoted in this proposition? (Ans. They must coincide.)

Exercises.

Prove the following:

1. The line that bisects the vertical angle of an isosceles triangle bisects the base perpendicularly.

2. If two adjacent sides of a quadrilateral are equal and the diagonal bisects the angle between them, then their remaining sides are equal.

3. If two segments stand perpendicularly to each other and if each bisects the other, then any point in either segment is equally distant from the endpoints of the other segment.

4. If equilateral triangles are constructed on the sides of any triangle, the distances between the vertices of the original triangle and the opposite vertices of the equilateral triangles are equal. (This should be proven after studying [1.32].)

PROPOSITION 1.5. *ISOSCELES TRIANGLES I. Suppose a given triangle is isosceles. Then*

1) if the sides of the triangle other than the base are extended, the angles under the base are equal to each other,

2) the angles at the base are equal to each other.

PROOF. Construct the triangle $\triangle ABC$ such that sides $AB = AC$ and denote side BC as the triangle's base. Extend the side AB to the segment BD and the side AC to the segment CE. We claim that the angles at the base ($\angle ABC$, $\angle ACB$) are equal in measure to one another and that the external angles below the base ($\angle DBC$, $\angle ECB$) are also equal in measure.

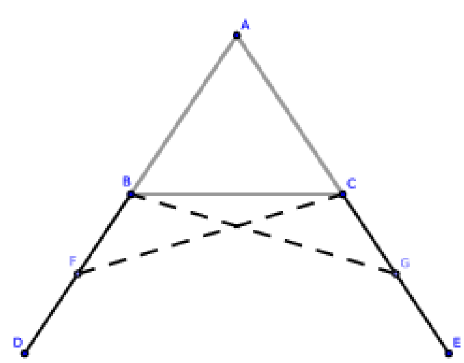

FIGURE 1.5.6. [1.5]

We prove each claim separately:

1. Let F be any point on the segment BD except for its endpoints. From AE, choose a point G such that $CG = BF$ [1.3]. Join BG, CF [Postulate 1.1].

Because $AF = AG$ by construction and $AC = AB$ by hypothesis, sides AF, AC in triangle $\triangle FAC$ are equal in length to sides AG, AB in triangle $\triangle GAB$. Also, the angle $\angle BAC$ is the interior angle to both pairs of sides in each triangle. By [1.4], $\triangle FAC \cong \triangle GAB$.

Again, because $AF = AG$ by construction and $AB = AC$ by hypothesis, the remaining segments BF and CG are equal in length (or $BF = CG$) [Axiom 1.3].

Notice that $\angle AFC = \angle BFC$ and $\angle AGB = \angle CGB$. Since we have shown that $FB = CG$, $FC = GB$, and $\angle AFC = \angle AGB$, by [1.4] we have that triangles $\triangle FBC \cong \triangle GCB$. Thus,

$$\angle DBC = \angle FBC = \angle GCB = \angle ECB$$

which are the angles under the base.

2. We also have that $\angle FCB = \angle GBC$ and $\angle FCA = \angle GBA$ by the above. Since $\angle FCA = \angle FCB + \angle ACB$ and $\angle GBA = \angle GBC + \angle ABC$, we obtain $\angle ACB = \angle ABC$ which are the angles at the base. □

Observation: The great difficulty which beginners have with this proposition is due to the fact that the two triangles $\triangle ACF$, $\triangle ABG$ overlap each other. A teacher or tutor should graph these triangles separately and point out the corresponding parts: $AF = AG$, $AC = AB$, and $\angle FAC = \angle GAB$. By [1.4], we have that $\angle ACF = \angle ABG$, $\angle AFC = \angle AGB$. The student should also be shown how to apply one of the triangles to the other so as to bring them into coincidence.

COROLLARY. *1. Every equilateral triangle is equiangular.*

Exercises.

1. Prove that the angles at the base are equal without extending the sides. Do the same by extending the sides through the vertex.

2. Prove that the line joining the point A to the intersection of the segments CF and BG is an Axis of Symmetry of $\triangle ABC$.

3. If two isosceles triangles stand on the same base, either on the same or on opposite sides of it, the line joining their vertices is an Axis of Symmetry of the figure formed by them. (Hint: this follows almost immediately from the proof of #2.)

4. Show how to prove this proposition by assuming as an axiom that every angle has a bisector.

5. Prove that each diagonal of a rhombus is an Axis of Symmetry of the rhombus. (Hint: this follows almost immediately from the proof of #3.)

6. If three points are taken on the sides of an equilateral triangle (one on each side and at equal distances from the angles), then the segments joining them form a new equilateral triangle.

PROPOSITION 1.6. *ISOSCELES TRIANGLES II. If a given triangle has two equal angles, then the sides opposite the two angles are equal in length (i.e., the triangle is isosceles).*

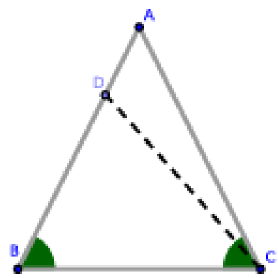

FIGURE 1.5.7. [1.6]

PROOF. Let $\triangle ABC$ be a triangle such that $\angle ABC = \angle ACB$. To show that side $AB = AC$, we will use a proof by contradiction[9].

Without loss of generality[10], suppose side $AB > AC$. On AB, construct a point D such that $BD = CA$ [1.3] and join points C and D [Axiom 1.1]. Then in triangles $\triangle DBC$, $\triangle ACB$, we have that $BD = AC$ with BC being a common side to both. Therefore, the two sides DB, BC in triangle $\triangle DBC$ are equal to the two sides AC, CB in $\triangle ACB$. Also, we have that $\angle DBC = \angle ABC$ by hypothesis. By [1.4], we have that $\triangle DBC \cong \triangle ACB$, a contradiction, since $AB = AD + DB$. It follows that AC, AB are not unequal; that is, $AC = AB$. □

Examination questions.

1. What is the hypothesis in this proposition?

2. What proposition is this the converse of?

[9]That is, we will show that the statement "A triangle with two equal angles has unequal opposite sides" is false. See also https://en.wikipedia.org/wiki/Proof_by_contradiction

[10]This term is used before an assumption in a proof which narrows the premise to some special case; it is implied that either the proof for that case can be easily applied to all others or that all other cases are equivalent. Thus, given a proof of the conclusion in the special case, it is trivial to adapt it to prove the conclusion in all other cases. http://en.wikipedia.org/wiki/Without_loss_of_generality

3. What is the inverse of this proposition?
4. What is the inverse of [1.5]?
5. What is meant by an indirect proof? (Ans. A proof by contradiction.)
6. How does Euclid generally prove converse propositions?
7. What false assumption is made in order to prove the proposition?
8. What does this false assumption lead to?

COROLLARY. *1. A triangle is isosceles if and only if the angles at its base are equal.*

PROPOSITION 1.7. *UNIQUENESS OF TRIANGLES. Suppose that we have two distinct triangles which share a common base. Also suppose that at one endpoint of the base we have that the two sides which connect to this vertex are equal in length. It follows that the lengths of the sides of the triangles which are connected to the other endpoint of the base are unequal in length.*

PROOF. Construct distinct triangles $\triangle ADB$, $\triangle ACB$ which share the base AB. Suppose that $AC = AD$ where each side shares the common endpoint A. We claim that $BC \neq BD$.

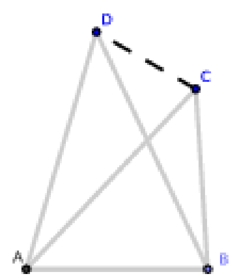

FIGURE 1.5.8. [1.7], case 1

We prove this claim in two cases.

Case 1: Let the vertex of each triangle lie outside the interior of the other triangle (i.e., such that D does not lie inside $\triangle ACB$ and C does not lie inside $\triangle ADB$). Join CD. Because $AD = AC$ by hypothesis, $\triangle ACD$ is isosceles. By [1.5], $\angle ACD = \angle ADC$.

However, $\angle ADC > \angle BDC$ since $\angle ADC = \angle ADB + \angle BDC$ [Axiom 1.9]. Therefore $\angle ACD > \angle BDC$. Since $\angle BCD = \angle BCA + \angle ACD$, we also have that $\angle BCD > \angle BDC$.

Now if $BD = BC$, we would have that $\angle BCD = \angle BDC$ [1.5]; however, we have shown that $\angle BCD \neq \angle BDC$. Hence, $BD \neq BC$.

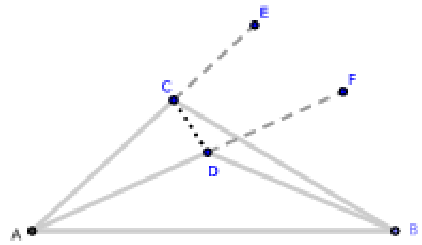

FIGURE 1.5.9. [1.7], case 2

Case 2: Wlog[11], let the vertex of the triangle $\triangle ADB$ fall within the interior of $\triangle ACB$. Extend side AC to segment CE and side AD to segment DF. Because $AC = AD$ by hypothesis, the triangle $\triangle ACD$ is isosceles, and by [1.5] $\angle ECD = \angle FDC$ (i.e., the external angles at the other side of the base CD are equal).

Notice that $\angle ECD > \angle BCD$ since $\angle ECD = \angle ECB + \angle BCD$ [Axiom 1.9]. Therefore $\angle FDC > \angle BCD$. Since $\angle BDC = \angle BDF + \angle FDC$, we have that $\angle BDC > \angle BCD$.

Now if $BC = BD$, we would have that $\angle BDC = \angle BCD$ [1.5]; however, we have shown that $\angle BCD \neq \angle BDC$. Hence $BD \neq BC$. □

COROLLARY. *1.7.1. Triangles which have no sides of equal length are distinct.*

Examination questions.

1. What use is made of [1.7]? (Ans: As a lemma to [1.8].)

2. In the demonstration of [1.7], the contrapositive of [1.5] occurs. Show where.

PROPOSITION 1.8. *THE "SIDE-SIDE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If three pairs of sides of two triangles are equal in length, then the triangles are congruent.*

PROOF. If two distinct triangles ($\triangle ABC$, $\triangle DEF$) have two sides (AB, AC) that are respectively equal to two sides of the other (DE, DF) where the base of one triangle (BC) equals the base of the other (EF), we claim that the two triangles are congruent.

[11]An abbreviation for "without loss of generality". http://en.wikipedia.org/wiki/Without_loss_of_generality

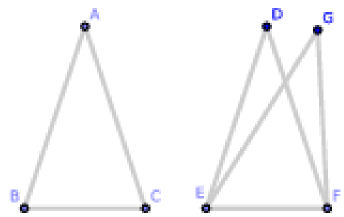

FIGURE 1.5.10. [1.8]

Let the triangle $\triangle ABC$ be applied to $\triangle DEF$ such that point B coincides with E and the side BC coincides with the side EF. Because $BC = EF$, the point C coincides with point F. If the vertex A falls on the same side of EF as vertex D, then the point A must coincide with D.

Otherwise, it must take a different position, such as point G. We then have $EG = BA$ and $BA = ED$ (by hypothesis). By [Axiom 1.1], $EG = ED$. Similarly, $FG = FD$, a contradiction since the triangles are distinct [1.7]. Hence the point A must coincide with D, and so the three angles of one triangle are respectively equal to the three angles of the other (specifically, $\angle ABC = \angle DEF$, $\angle BAC = \angle EDF$, and $\angle BCA = \angle EFD$). Therefore, $\triangle ABC \cong \triangle DEF$. $\square$

This proposition is the converse of [1.4] and is the second case of the congruence of triangles in the Elements.

Philo's Proof:

PROOF. Let the equal bases be applied as in the foregoing proof, but let the vertices fall on the opposite sides of the base. Let $\triangle BGC$ be a "copy" of $\triangle EDF$. Join AG. Because $BG = BA$, we have that $\angle BAG = \angle BGA$. Similarly, $\angle CAG = \angle CGA$.

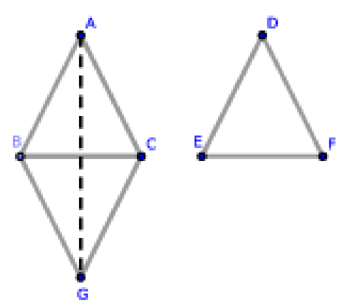

FIGURE 1.5.11. Philo's Proof of [1.8]

Notice that

$$\begin{array}{ccc} \angle BGA + \angle CGA & = & \angle BGC \\ \parallel & & \\ \angle BAG + \angle CAG & = & \angle BAC \end{array}$$

Hence, $\angle BAC = \angle BGC$. Since $\triangle BGC = \triangle EDF$ by construction, we have that $\triangle BAC = \triangle EDF$.

□

PROPOSITION 1.9. *BISECTING A RECTILINEAR ANGLE. It is possible to bisect an angle.*

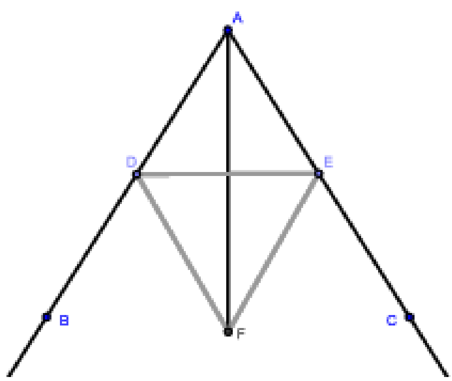

FIGURE 1.5.12. [1.9]

PROOF. Take any point D on the ray AB. Take the point E on AC such that $AE = AD$ [1.3]. Join DE [Postulate 1.1] and, on the opposite side of point A, construct the equilateral triangle $\triangle DEF$ [1.1]. Join AF. We claim that AF bisects $\angle BAC$.

Notice that triangles $\triangle DAF$ [$\triangle EAF$ share the side AF. Given that $AD = AE$ by construction, we have that the two sides DA, AF are respectively equal to the two sides EA, AF. Also, $DF = EF$ because they are the sides of an equilateral triangle [Def. 1.21].

By [1.8], $\angle DAF = \angle EAF$. Since $\angle BAC = \angle DAF + \angle EAF$, $\angle BAC$ is bisected by AF.

□

COROLLARY. *1. The line AF is an Axis of Symmetry of the figure $ABCF$, $\triangle AED$, and segment DE.*

COROLLARY. *2. In [1.9], AB and AC may be constructed as either lines, rays, or segments with point A as the vertex, mutatis mutandis.*

Examination questions.

1. Why does Euclid construct the equilateral triangle on the side opposite of A?

2. If the equilateral triangle were constructed on the other side of DE, in what case would the construction fail?

Exercises.

1. Prove [1.9] without using [1.8]. (Hint: the proof follows almost immediately from [1.5, #2].)

2. Prove that $AF \perp DE$. (Hint: the proof follows almost immediately from [1.5, #2].)

3. Prove that any point on AF is equally distant from the points D and E.

PROPOSITION 1.10. *BISECTING A STRAIGHT-LINE SEGMENT. It is possible to bisect an arbitrary segment; that is, it is possible to locate the midpoint of a segment.*

PROOF. We wish to bisect the segment AB.

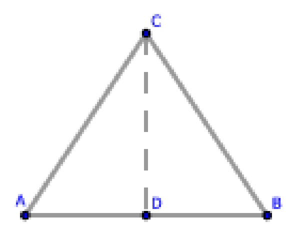

FIGURE 1.5.13. [1.10]

Construct an equilateral triangle $\triangle ACB$ on segment AB [1.1]. Bisect $\angle ACB$ by the segment CD [1.9], intersecting AB at point D. We claim that AB is bisected at D.

The two triangles $\triangle ACD$, $\triangle BCD$ have sides AC, BC such that $AC = BC$ (since each are sides of an equilateral triangle) and also share side CD in common. Therefore, the two sides AC, CD in one triangle are equal to the two sides BC, CD in the other. We also have that $\angle ACD = \angle BCD$ by construction. By [1.4], we have that $AD = DB$. Since $AB = AD + DB$, it follows that AB is bisected at D. □

Exercises.

1. Bisect a segment by constructing two circles.

2. Extend CD to a line. Prove that every point equally distant from the points A, B are points in the line CD.

PROPOSITION 1.11. *CONSTRUCTING A PERPENDICULAR STRAIGHT-LINE SEGMENT TO A LINE I. It is possible to construct a segment at a right angle to a given line from an arbitrary point on the line.*

PROOF. Construct the line AB containing the point C. We wish to construct the segment CF such that $AB \perp CF$.

On AC, take any point D and choose some point E on AC such that $CE = CD$ [1.3]. Construct the equilateral triangle $\triangle DFE$ on DE [1.1] and join CF. We claim that $AB \perp CF$.

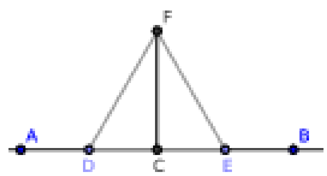

FIGURE 1.5.14. [1.11]

The two triangles $\triangle DCF$, $\triangle ECF$ have $CD = CE$ by construction and CF in common; therefore, the two sides CD, CF in one triangle are respectively equal to the two sides CE, CF in the other, and $DF = EF$ since they are the sides of an equilateral triangle [Def. 1.21]. By [1.8], $\angle DCF = \angle ECF$. Since these are adjacent angles, by [Def. 1.13] each is a right angle, and so $AB \perp CF$ at point C. □

COROLLARY. *1. [1.11] holds when* **AB** *is a segment or ray and/or when* **CF** *is a straight line or a ray, mutatis mutandis.*

Exercises.

1. Prove that the diagonals of a lozenge bisect each other perpendicularly.

2. Prove [1.11] without using [1.8]. (Hint: The proof follows from the result of [1.9, #2].)

3. Find a point on a given line that is equally distant from two given points.

4. Find a point on a given line such that if it is joined to two given points on opposite sides of the line, then the angle formed by the connecting segment is bisected by the given line. (Hint: similar to the proof of #3.)

5. Find a point that is equidistant from three given points. (Hint: you are looking for the circumcenter of the triangle formed by the points[12].)

PROPOSITION 1.12. *CONSTRUCTING A PERPENDICULAR STRAIGHT-LINE SEGMENT TO A LINE II. Given an arbitrary straight line and an arbitrary point not on the line, we may construct a perpendicular segment from the point to the line.*

PROOF. We wish to construct a perpendicular segment to a given line AB from a given point C which is not on AB.

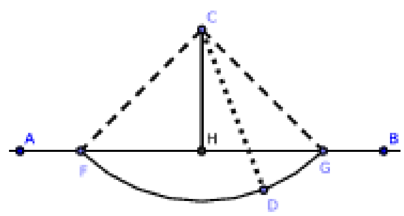

FIGURE 1.5.15. [1.12]

Take any point D which is not on AB and stands on the opposite side of AB from C. Construct the circle $\circ FDG$ with C as its center and CD as radius [Postulate 1.3] which intersects AB at the points F and G. Bisect the segment FG at H [1.10] and join CH [Postulate 1.1]. We claim that $CH \perp AB$.

To see this, join CF, CG. Then the two triangles $\triangle FHC$, $\triangle GHC$ have sides $FH = GH$ by construction and side HC in common. We also have that $CF = CG$ since both are radii of $\circ FDG$ [Def. 1.32]. Therefore, $\angle CHF = \angle CHG$ [1.8]. Since these are adjacent angles, by [Def. 1.13] each is a right angle, and so $CH \perp AB$. □

COROLLARY. *1. [1.12] holds when CH and/or AB are replaced by rays, mutatis mutandis.*

12 http://www.mathopenref.com/trianglecircumcenter.html

Exercises.

1. Prove that circle $\circ FDG$ cannot meet AB at more than two points.

2. If one angle of a triangle is equal to the sum of the other two, prove that the triangle can be divided into the sum of two isosceles triangles and that the length of the base is equal to twice the segment from its midpoint to the opposite angle.

PROPOSITION 1.13. *ANGLES AT INTERSECTIONS OF STRAIGHT LINES. If a line intersects another line at one and only one point, the lines will either make two right angles or two angles whose sum equals two right angles.*

PROOF. If the line AB intersects the line CD at one and only one point (B), we claim that either $\angle ABC$ and $\angle ABD$ are right angles or the sum $\angle ABC + \angle ABD$ equals two right angles.

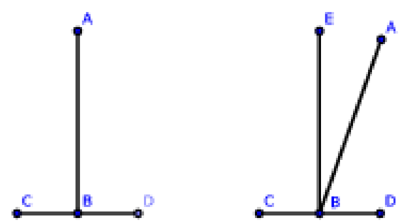

FIGURE 1.5.16. [1.13] (α), (β)

If $AB \perp CD$ as in Fig. 1.5.16(α), then $\angle ABC$ and $\angle ABD$ are right angles.

Otherwise, construct $BE \perp CD$ [1.11]. Notice that $\angle CBA = \angle CBE + \angle EBA$ [Def. 1.11]. Adding the measure of $\angle ABD$ to each side of the equality, we obtain that

$$\angle CBA + \angle ABD = \angle CBE + \angle EBA + \angle ABD$$

Similarly, we have that

$$\angle CBE + \angle EBD = \angle CBE + \angle EBA + \angle ABD$$

Since quantities which are equal to the same quantity are equal to one another, we have that

$$\angle CBA + \angle ABD = \angle CBE + \angle EBD$$

Since $\angle CBE$, $\angle EBD$ are right angles, we have that $\angle CBA + \angle ABD$ equals the sum of two right angles. $\square$

An alternate proof:

PROOF. Denote $\angle EBA$ by θ. We then have that

$$\begin{aligned} \angle CBA &= \text{right angle } + \theta \\ \angle ABD &= \text{right angle } - \theta \quad \Rightarrow \\ \angle CBA + \angle ABD &= \text{right angle} \end{aligned}$$

□

COROLLARY. *1. The above proposition holds when the straight lines are replaced by segments and/or rays, mutatis mutandis.*

COROLLARY. *2. The sum of two supplemental angles equals two right angles.*

COROLLARY. *3. Two distinct straight lines cannot share a common segment.*

COROLLARY. *4. The bisector of any angle bisects the corresponding reentrant angle.*

COROLLARY. *5. The bisectors of two supplemental angles are at right angles to each other.*

COROLLARY. *6. The angle $\angle EBA$ is half the difference of the angles $\angle CBA$, $\angle ABD$.*

PROPOSITION 1.14. *RAYS TO STRAIGHT LINES. If at the endpoint of a ray there exists two other rays standing on opposite sides of that ray such that the sum of their adjacent angles is equal to two right angles, then these two rays form one line.*

PROOF. Construct the ray BA with endpoint B. Suppose at B there exists two other rays BC and BD which stand on opposite sides of BA such that the sum of their adjacent angles, $\angle CBA + \angle ABD$, equals two right angles. We claim that $BC \oplus BD = CD$ where CD is a line.

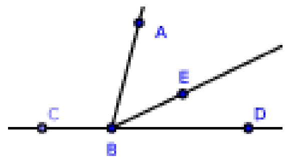

FIGURE 1.5.17. [1.14]

Suppose instead that the rays BC and BE form the straight line CE. Since CE is a line and BA stands on it, the sum $\angle CBA + \angle ABE$ equals two right angles [1.13]. Also by hypothesis, the sum $\angle CBA + \angle ABD$ equals two right angles. Therefore,

$$\angle CBA + \angle ABE = \angle CBA + \angle ABD$$

Removing the angle in common, $\angle CBA$, we have that $\angle ABE = \angle ABD$. However, $\angle ABD = \angle ABE + \angle EBD$, a contradiction [Axiom 1.9]. Hence, $BC \oplus BD = CD$ where CD is a line. □

COROLLARY. *1. If at a point on a straight line, segment, or ray, two segments on opposite sides of the line make the sum of the adjacent angles equal to two right angles, these two segments also form a single segment.*

PROPOSITION 1.15. *OPPOSITE ANGLES ARE EQUAL. If two lines intersect one another at one point, their opposite angles are equal.*

PROOF. If two straight lines AB, CD intersect one another at one point, E, then their opposite angles are equal ($\angle CEA = \angle DEB$ and $\angle BEC = \angle AED$).

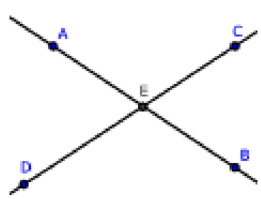

FIGURE 1.5.18. [1.15]

Because the line AB intersects CD at E, the sum $\angle CEA + \angle AED$ equals two right angles [1.13]. Because the line CD intersects AB at the point E, the sum $\angle CEA + \angle BEC$ also equals two right angles. Therefore,

$$\angle CEA + \angle AED = \angle CEA + \angle BEC$$

Removing the common angle $\angle CEA$, we have that $\angle AED = \angle BEC$. By an analogous method, we also obtain that $\angle CEA = \angle DEB$. □

An alternate proof:

PROOF. Because opposite angles share a common supplement, they are equal. □

COROLLARY. *1. [1.15] holds when either one or both of the two straight lines are replaced either by segments or by rays, mutatis mutandis.*

Examination questions for [1.13]-[1.15].

1. What problem is required in Euclid's proof of [1.13]?
2. What theorem? (Ans. No theorem, only the axioms.)
3. If two lines intersect, how many pairs of supplemental angles do they make?
4. What is the relationship between [1.13] and [1.14]?
5. What three lines in [1.14] are concurrent?
6. What caution must be taken as we prove [1.14]?
7. State the converse of Proposition [1.15] and prove it.
8. What is the subject of [1.13], [1.14], [1.15]? (Ans. Angles at a point.)

PROPOSITION 1.16. *THE EXTERIOR ANGLE IS GREATER THAN EITHER OF THE NON-ADJACENT INTERIOR ANGLES. If any side of a triangle is extended, the exterior angle is greater than either of the non-adjacent interior angles.*

PROOF. Construct $\triangle ABC$. Wlog, we extend side BC to the segment CD. We claim that the exterior angle $\angle ACD$ is greater than either of the interior non-adjacent angles $\angle ABC$, $\angle BAC$.

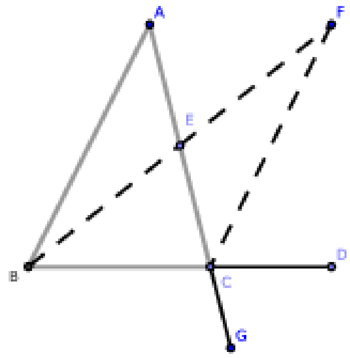

FIGURE 1.5.19. [1.16]

Bisect AC at point E [1.10] and join BE [Postulate 1.1]. Extend BE to EF such that $BE = EF$ [1.3]. Join CF. Because $EC = EA$ by construction, the triangles $\triangle CEF$, $\triangle AEB$ have the sides CE, EF in one equal to the sides AE, EB in the other. We also have that $\angle CEF = \angle AEB$ [1.15]. By [1.4], the angle $\angle ECF = \angle EAB$. However, $\angle ACD$ is greater than $\angle ECF$ since $\angle ACD = \angle ECF + \angle FCD$. Therefore, the angle $\angle ACD > \angle EAB = \angle BAC$.

Similarly, it can be shown that if side AC is extended to segment CG, then the exterior angle $\angle BCG > \angle ABC$. But $\angle BCG = \angle ACD$ [1.15]. Hence, $\angle ACD$ is greater than either of the interior non-adjacent angles $\angle ABC$ or $\angle BAC$. □

COROLLARY. *1. The sum of the three interior angles of the triangle* $\triangle BCF$ *is equal to the sum of the three interior angles of the triangle* $\triangle ABC$.

COROLLARY. *2. The area of* $\triangle BCF$ *is equal to the area of* $\triangle ABC$*, which we will write as* $\triangle BCF = \triangle ABC$.

COROLLARY. *3. The lines* BA *and* CF*, if extended, cannot meet at any finite distance. For, if they met at any finite point* X*, the triangle* $\triangle CAX$ *would have an exterior angle* $\angle BAC$ *equal to the interior angle* $\angle ACX$.

Exercise.

1. Prove [1.16, Cor. 3] using a proof by contradiction.

PROPOSITION 1.17. *THE SUM OF TWO ANGLES OF A TRIANGLE. The sum of two angles of a triangle is less than that of two right angles.*

PROOF. We claim that the sum of any two angles of a triangle $\triangle ABC$ is less than the sum of two right angles (wlog, we choose $\angle ABC$ and $\angle BAC$).

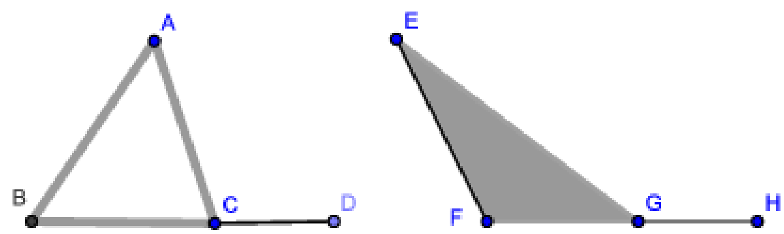

FIGURE 1.5.20. [1.17]

Extend side BC to the segment CD. By [1.16], the exterior angle $\angle ACD$ is greater than $\angle ABC$. To each, add the angle $\angle ACB$, and we obtain $\angle ACD + \angle ACB > \angle ABC + \angle ACB$. However, $\angle ACD + \angle ACB$ equals two right angles [1.13]. Therefore, $\angle ABC + \angle ACB$ is less than two right angles.

Similarly, we may show that the sums $\angle ABC + \angle BAC$ and $\angle ACB + \angle BAC$ are each less than two right angles.

A similar argument follows on $\triangle EFG$, *mutatis mutandis*. □

COROLLARY. *1. Every triangle has at least two acute angles.*

COROLLARY. *2. If two angles of a triangle are unequal, the lesser is acute.*

Exercise.

1. Prove [1.17] without extending a side. (Attempt after completing Chapter 1. Hint: use parallel line theorems.)

PROPOSITION 1.18. *ANGLES AND SIDES IN A TRIANGLE I. In a triangle, if one side is longer than another, then the angle opposite to the longer side is greater in measure than the angle opposite to the shorter side.*

PROOF. Suppose we have $\triangle ABC$ with sides AB, AC such that $AC > AB$. We claim that the angle opposite AC ($\angle ABC$) is greater in measure than the angle opposite AB ($\angle ACB$).

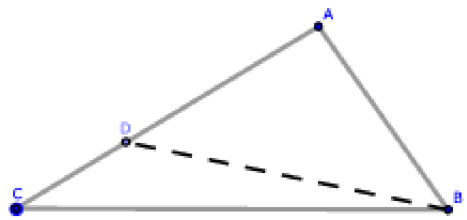

FIGURE 1.5.21. [1.18]

From AC, cut off AD such that $AD = AB$ [1.3]. Join BD [Postulate 1.1]. It follows that $\triangle ABD$ is isosceles; therefore, $\angle ADB = \angle ABD$. Now $\angle ADB > \angle ACB$ [1.16], and so $\angle ABD > \angle ACB$. Since $\angle ABC = \angle ABD + \angle CBD$, we also have that $\angle ABC > \angle ABD$ from which it follows that $\angle ABC > \angle ACB$. □

An alternate proof:

PROOF. With A as the center and with the shorter side AB as radius, construct the circle $\circ BED$ which intersects BC at point E.

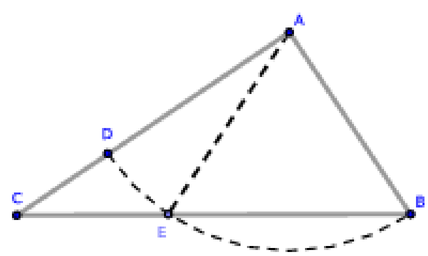

FIGURE 1.5.22. [1.18], alternate proof

Join AE. Since $AB = AE$, $\angle AEB = \angle ABE$; however, $\angle AEB > \angle ACB$ [1.16]. Therefore, $\angle ABE > \angle ACB$. □

Exercises.

1. Prove that if two of the opposite sides of a quadrilateral are respectively the greatest and the least sides of the quadrilateral, then the angles adjacent to the least are greater than their opposite angles.

2. In any triangle, prove that the perpendicular from the vertex opposite the side which is not less than either of the remaining sides falls within the triangle.

PROPOSITION 1.19. *ANGLES AND SIDES IN A TRIANGLE II. In a triangle, if one angle is greater in measure than another, then the side opposite the greater angle is longer than the side opposite the shorter angle.*

PROOF. Construct $\triangle ABC$ with sides AB, AC. We claim that if $\angle ABC > \angle ACB$, then $AC > AB$.

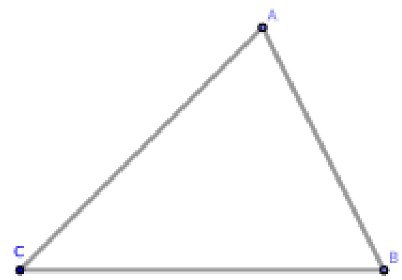

FIGURE 1.5.23. [1.19]

If AC is not longer than AB, then either $AC = AB$ or $AC < AB$.

1. If $AC = AB$, $\triangle ACB$ is isosceles and $\angle ACB = \angle ABC$ [1.5]. This contradicts our hypothesis, and so $AC \neq AB$.

2. If $AC < AB$, we have that $\angle ACB < \angle ABC$ [1.18]. This also contradicts our hypothesis, and so $AC \not< AB$.

Since $AC \not\leq AB$, we must have that $AC > AB$. □

COROLLARY. *1. In a triangle, greater (lesser) sides stand opposite the greater (lesser) angles and greater (lesser) angles stand opposite the greater (lesser) sides.*

Exercises.

1. Prove this proposition by a direct demonstration.

2. Prove that a segment from the vertex of an isosceles triangle to any point on the base is less than either of the equal sides but greater if the base is extended and the point of intersection falls outside of the triangle.

3. Prove that three equal segments cannot be constructed from the same point to the same line.

4. If in [1.16], Fig 1.5.19, AB is the longest side of the $\triangle ABC$, then BF is the longest side of $\triangle FBC$ and $\angle BFC$ is less than half of $\angle ABC$.

5. If $\triangle ABC$ is a triangle such that side $AB \not< AC$, then a segment AG, constructed from A to any point G on side BC, is less than AC.

PROPOSITION 1.20. *THE SUM OF THE LENGTHS OF ANY PAIR OF SIDES OF A TRIANGLE. In a triangle, the sum of the lengths of any pair of sides is greater than the length of the remaining side.*

PROOF. We claim that the sum of any two sides (BA, AC) of a triangle $\triangle ABC$ is greater than the third (BC).

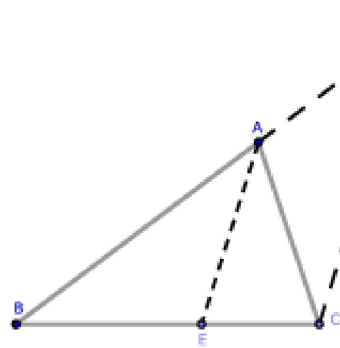

FIGURE 1.5.24. [3.20]

Extend BA to the segment AD [Postulate 1.2] such that $AD = AC$ [1.3]. Join CD, constructing $\triangle ACD$. Because $AD = AC$, $\angle ACD = \angle ADC$ [1.5]. Since $\angle BCD = \angle BCA + \angle ACD$, $\angle BCD > \angle ADC = \angle BDC$. It follows that the side $BD > BC$ [1.19].

Noticing that

$$\begin{aligned} AD &= AC && \text{and} \\ BA + AD &= BA + AC && \Rightarrow \\ BD &= BA + AC \end{aligned}$$

we obtain that $BA + AC > BC$. □

Alternatively:

PROOF. Bisect the angle $\angle BAC$ by AE [1.9]. Then the angle $\angle BEA$ is greater than $\angle EAC$. However, $\angle EAC = \angle EAB$ by construction. Therefore, the angle $\angle BEA > \angle EAB$. It follows that $BA > BE$ [1.19]. Similarly, $AC > EC$. It follows that $BA + AC > BE + EC = BC$. □

Exercises.

1. In any triangle, the difference between the lengths of any two sides is less than the length of the third.
2. Any side of any polygon is less than the sum of the remaining sides.

3. The perimeter of any triangle is greater than that of any inscribed triangle and less than that of any circumscribed triangle. (See also [Def. 4.1].)

4. The perimeter of any polygon is greater than that of any inscribed (and less than that of any circumscribed) polygon of the same number of sides.

5. The perimeter of a quadrilateral is greater than the sum of its diagonals.

6. The sum of the lengths of the three medians of a triangle is less than 3/2 times its perimeter.

PROPOSITION 1.21. *TRIANGLES WITHIN TRIANGLES. In an arbitrary triangle, if two segments are constructed from the vertexes of its base to a point within the triangle, then*

1) the sum of these inner sides will be less than the sum of the outer corresponding sides (i.e., the outer sides excluding the base);

2) these inner sides will contain a greater angle than the corresponding sides of the outer triangle.

PROOF. If two segments (BD, CD) are constructed to a point (D) within a triangle ($\triangle ABC$) from the endpoints of its base (BC), we claim that:

1. $BA + AC > BD + DC$

2. $\angle BDC > \angle BAC$

Construct $\triangle ABC$ and $\triangle BDC$ as in Fig. 1.5.25(α).

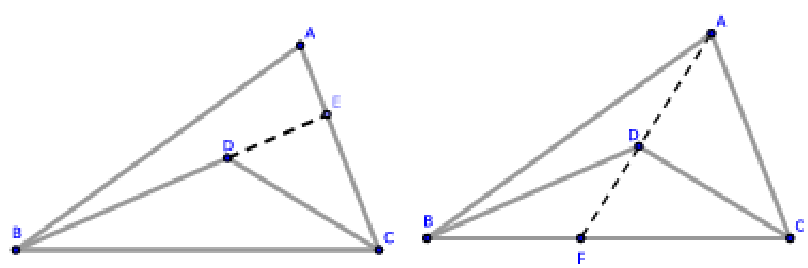

FIGURE 1.5.25. [3.21] (α), (β)

1. Extend BD to meet AC at point E [Postulate 1.2]. In triangle $\triangle BAE$, we have that $BA + AE > BE$ [1.20], from which it follows that

$$BA + AC = BA + AE + EC > BE + EC$$

Similarly, in $\triangle DEC$, we have that $DE + EC > DC$, from which it follows that

$$BE + EC = BD + DE + EC > BD + DC$$

From these two inequalities, we obtain that $BA + AC > BD + DC$.

2. Consider $\triangle DEC$. By [1.16], we have that $\angle BDC > \angle BEC$. Similarly in $\triangle ABE$, $\angle BEC > \angle BAE$. It follows that $\angle BDC > \angle BAE = \angle BAC$. □

An alternative proof to part 2 that does not extend sides BD, DC:

PROOF. Construct $\triangle ABC$ and $\triangle BDC$ as in Fig. 1.5.25(β), joining AD and extending it to meet BC at point F. Consider $\triangle BDA$ and $\triangle CDA$. By [1.16], $\angle BDF > \angle BAF$ and $\angle FDC > \angle FAC$. Since

$$\begin{aligned}\angle BDF + \angle FDC &= \angle BDC \\ \angle BAF + \angle FAC &= \angle BAC\end{aligned}$$

we have that $\angle BDC > \angle BAC$. □

Exercises.

1. The sum of the side lengths constructed from any point within a triangle to its angular points is less than the length of the triangle's perimeter.

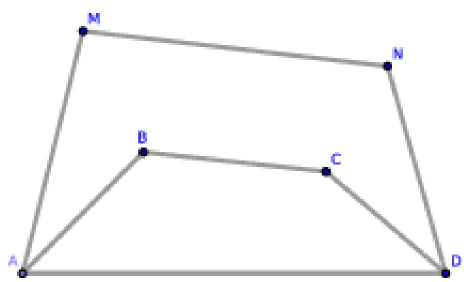

FIGURE 1.5.26. [1.21, #2]

2. If a convex polygonal line $ABCD$ lies within a convex polygonal line $AMND$ terminating at the same endpoints, prove that the length of the former is less than that of the latter.

PROPOSITION 1.22. *CONSTRUCTION OF TRIANGLES FROM ARBITRARY SEGMENTS. It is possible to construct a triangle whose three sides are respectively equal to three arbitrary segments whenever the sum of every two pairs of segments is greater than the length of the remaining segment.*

PROOF. Let AR, BS, and CT be arbitrary segments which satisfy our hypothesis.

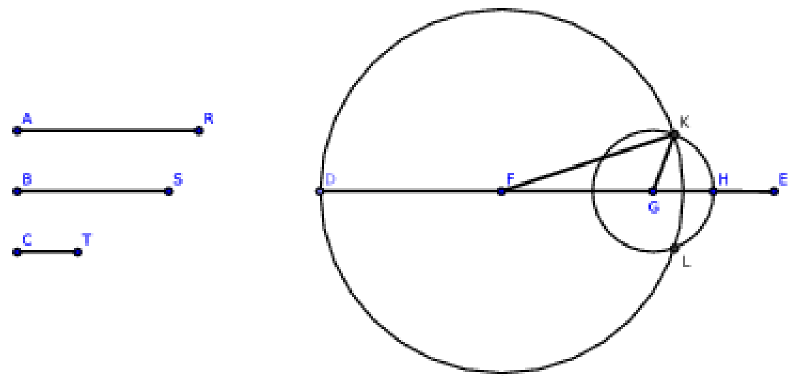

FIGURE 1.5.27. [1.22]

Take the ray DE and cut off the segments $DF = AR$, $FG = BS$, and $GH = CT$ [1.3]. With F as the center and FD as radius, construct the circle $\circ KDL$ [Postulate 1.3]. With G as the center and GH as radius, construct the circle $\circ KHL$ where K is a point of intersection between the circles $\circ KDL$ and $\circ KHL$. Join KF, KG. We claim that $\triangle KFG$ is the required triangle.

Since F is the center of $\circ KDL$, $FK = FD$. Since $FD = AR$ by construction, $FK = AR$ [Axiom 1.1]. Also by construction, we have that $GK = CT$ and $FG = BS$. Hence, the three sides of the triangle $\triangle KFG$ are respectively equal to the three segments AR, BS, and CT. □

Examination questions.

1. What is the reason for our condition that the sum of every two of the given segments must be greater than the length of the third?

2. Under what conditions would the circles fail to intersect?

Exercises.

1. Prove that when the above condition is fulfilled that the two circles must intersect.

2. If the sum of two of the segments equals the length of the third, would the circles meet? Prove that they would not intersect.

PROPOSITION 1.23. *CONSTRUCTING AN ANGLE EQUAL TO AN ARBITRARY RECTILINEAR ANGLE. It is possible to construct an angle equal to an arbitrary angle on an endpoint of a segment.*

PROOF. Construct an arbitrary angle $\angle DEF$ from the rays ED, EF. From a given point A on a given segment AB, we wish to construct an angle equal to $\angle DEF$.

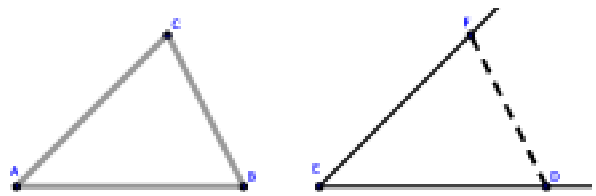

FIGURE 1.5.28. [1.23]

Join DF and construct the triangle $\triangle BAC$ whose sides are respectively equal to those of DEF; specifically, let $AB = ED$, $AC = EF$, and $CB = FD$ [1.22]. By [1.8], $\angle BAC = \angle DEF$. □

Exercises.

1. Construct a triangle given two sides and the angle between them.
2. Construct a triangle given two angles and the side between them.
3. Construct a triangle given two sides and the angle opposite one of them.
4. Construct a triangle given the base, one of the angles at the base, and the sum or difference of the sides.
5. Given two points, one of which is in a given line, find another point on the given line such that the sum or difference of its distances from the former points may be given. Show that two such points may be found in each case.

PROPOSITION 1.24. *ANGLES AND SIDES IN A TRIANGLE III. If in two triangles we have two pairs of sides in each triangle respectively equal to the other where the interior angle in one triangle is greater in measure than the interior angle of the other triangle, then the remaining sides of the triangles will be unequal in length; specifically, the triangle with the greater interior angle will have a greater side than the triangle with the lesser interior angle.*

PROOF. Construct two triangles $\triangle ABC$, $\triangle DEF$ where two sides of one (AB, AC) are respectively equal to two sides of the other (DE, DF) but the interior angle of $\triangle ABC$ ($\angle BAC$) is greater than the interior angle of the $\triangle DEF$ ($\angle EDF$). We claim that the base of $\triangle ABC$ (BC) is longer than the base of $\triangle DEF$ (EF). (Or, if $AB = DE$, $AC = DF$, and $\angle BAC > \angle EDF$, then $BC > EF$.)

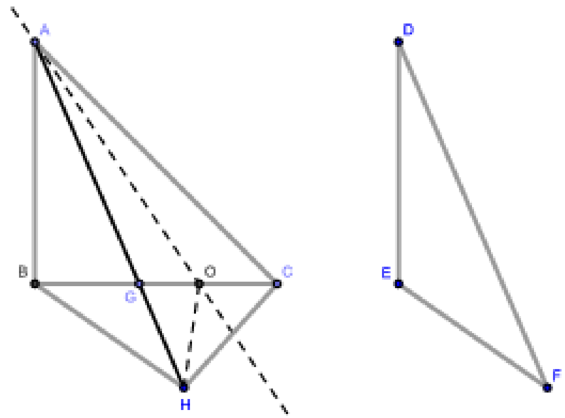

FIGURE 1.5.29. [1.24]

Construct point G on AC such that $\angle BAG = \angle EDF$. Wlog, suppose that $AC > AB$. Because $AB \not> AC$, $AG < AC$ [1.19, #5]. Extend AG to point H where $AH = DF = AC$ [1.3].

Join BH. In triangles $\triangle BAH$, $\triangle EDF$, we have $AB = DE$ by hypothesis, and and $\angle BAH = \angle EDF$ by construction. By [1.4], it follows that $BH = EF$.

Again, because $AH = AC$ by construction, the triangle $\triangle ACH$ is isosceles; therefore, $\angle ACH = \angle AHC$ [1.5]. However, $\angle ACH > \angle BCH$ since $\angle ACH = \angle BCH + \angle BCA$. It follows that $\angle AHC > \angle BCH$. And since $\angle BHC = \angle BHA + \angle AHC$, we also have that $\angle BHC > \angle BCH$. By [1.19], the greater angle stands opposite to the longer side, and so $BC > BH$. Since $BH = EF$, it follows that $BC > EF$. □

Alternatively, the concluding part of this proposition may be proved without joining CH.

PROOF. Construct the triangles as above. We have that

$$\begin{array}{rcll} BG + GH & > & BH & [1.20] \quad \text{and} \\ AG + GC & > & AC & [1.20] \quad \Rightarrow \\ BC + AH & > & BH + AC & \end{array}$$

Since $AH = AC$ by construction, we have that $BC > BH = EF$. □

Another alternative:

PROOF. In $\triangle ABC$, bisect the angle $\angle CAH$ by AO and join OH. Now in $\triangle CAO$, $\triangle HAO$ we have the sides CA, AO in one triangle equal to the sides AH, AO in the other where the interior angles are equal. By [1.4], $OC = OH$.

It follows that $BO + OH = BO + OC = BC$. But $BO + OH > BH$ [1.20]. Therefore, $BC > BH = EF$. □

Exercises.

1. Prove this proposition by constructing the angle $\angle ABH$ to the left of AB.

2. Prove that the angle $\angle BCA > \angle EFD$.

PROPOSITION 1.25. *ANGLES AND SIDES IN A TRIANGLE IV. If in two triangles we have two pairs of sides in each triangle respectively equal to the other where the remaining sides of the triangles are unequal, the interior angle of one triangle will be greater in measure than that of the other triangle; specifically, the triangle with the longer side will have a greater interior angle than the triangle with the shorter side.*

PROOF. If two triangles ($\triangle ABC$, $\triangle DEF$) have two sides of one triangle (AB, AC) respectively equal to two sides of the other (DE, DF) where the base of one (BC) is greater than the base of the other (EF), the angle ($\angle BAC$) contained by the sides of the triangle with the longer base ($\triangle ABC$) is greater in measure than the angle ($\angle EDF$) contained by the sides of the other ($\triangle DEF$). (Or, if $AB = DE$, $AC = DF$, and $BC > EF$, then $\angle BAC > \angle EDF$.)

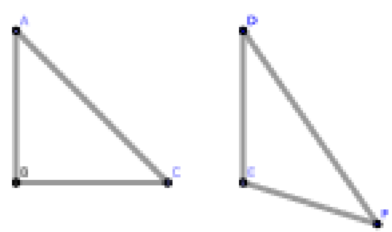

FIGURE 1.5.30. [1.25]

If $\angle BAC \not> \angle EDF$, then either $\angle BAC = \angle EDF$ or $\angle BAC < \angle EDF$. We divide the proof into two cases:

1. If $\angle BAC = \angle EDF$, then since the triangles $\triangle ABC$, $\triangle DEF$ have the two sides AB, AC of one respectively equal to the two sides DE, DF of the other, it follows that $BC = EF$ by [1.4], which contradicts our hypothesis that $BC > EF$. Hence, $\angle BAC \neq \angle EDF$.

2. If $\angle EDF > \angle BAC$, then because the triangles $\triangle DEF$, $\triangle ABC$ have the two sides DE, DF of one respectively equal to the two sides AB, AC of the

other, by [1.24], $EF > BC$, which contradicts our hypothesis that $BC > EF$. Therefore $\angle EDF \not> \angle BAC$.

Since it is not the case that either $\angle BAC = \angle EDF$ or $\angle BAC < \angle EDF$, we must have that $\angle BAC > \angle EDF$. □

An alternate proof:

PROOF. Construct $\triangle ABC$, $\triangle DEF$ as in the previous proof as well as the triangle $\triangle ACG$ where sides $AG = DE$, $GC = EF$, $CA = FD$ of the triangle $\triangle DEF$ [1.22]. Join BG. Because $BC > EF$ by hypothesis, it follows that $BC > GC$. By [1.18], $\angle BGC > \angle GBC$. Construct $\angle BGH = \angle GBH$ [1.23] and join AH. By [1.6], $BH = GH$.

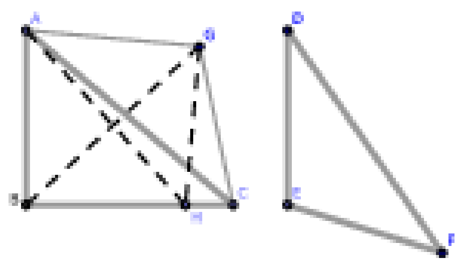

FIGURE 1.5.31. [1.25], alternate proof

Therefore in $\triangle ABH$, $\triangle AGH$ we have that $AB = AG$, $BH = GH$, and the side AH in common. By [1.8], $\angle BAH = \angle GAH$. Since $\angle BAC = \angle BAH + \angle CAH$, we also have that

$$\begin{aligned}\angle BAC &= \angle GAH + \angle CAH \\ &= \angle CAG + 2 \cdot \angle CAH\end{aligned}$$

and therefore $\angle BAC > \angle CAG$. By [1.8], we have that $\angle CAG = \angle EDF$, and so $\angle BAC > \angle EDF$. □

COROLLARY. *1. In two triangles with two pairs of sides respectively equal to the other, the final sides are unequal if and only if the interior angles of the two pairs of sides are also unequal. Specifically, the statement that "the triangle with the longer side contains a greater interior angle than the triangle with the shorter side" is equivalent to the statement that "the triangle with the greater interior angle has a longer base than the triangle with the lesser interior angle".*

Exercise.

1. Demonstrate this proposition directly by cutting off from BC a segment equal in length to EF.

PROPOSITION 1.26. *If in two triangles we have two pairs of angles in each triangle respectively equal to the other and one side in each triangle respectively equal to the other, then the triangles are congruent.*

1. THE "ANGLE-SIDE-ANGLE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If the side in question is the side between the two angles, then the triangles are congruent.

2. THE "ANGLE-ANGLE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If the side in question is not the side between the two angles, then the triangles are congruent.

PROOF. If two triangles ($\triangle ABC$, $\triangle DEF$) have two angles of one ($\angle ABC$, $\angle ACB$) respectively equal to two angles of the other ($\angle DEF$, $\angle EFD$) and a side of one equal to a similarly placed side of the other (placed with regard to the angles), then $\triangle ABC \cong \triangle DEF$.

This proposition breaks down into two cases according to whether the equal sides are adjacent or opposite to the equal angles. We prove each case separately:

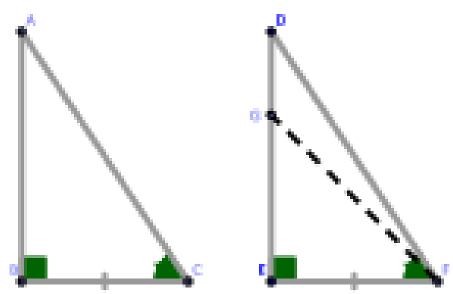

FIGURE 1.5.32. [1.26], case 1

1. Suppose that $BC = EF$. If $AB \neq DE$, suppose that $AB = GE$. Join GF. Then the triangles $\triangle ABC$, $\triangle GEF$ have the sides AB, BC of one respectively equal to the sides GE, EF of the other where$\angle ABC = \angle GEF$ by hypothesis. By [1.4], the angle $\angle ACB = \angle GFE$. Since $\angle ACB = \angle DFE$ by hypothesis, we have that $\angle GFE = \angle EFD$ and $\angle GFE + \angle GFD = \angle EFD$, a contradiction. Therefore AB and DE are not unequal, or $AB = DE$.

Consequently, the triangles $\triangle ABC$, $\triangle DEF$ have the sides AB, BC of one respectively equal to the sides DE, EF of the other where interior angles $\angle ABC$ and $\angle DEF$ are equal. By [1.4], $\triangle ABC \cong \triangle DEF$.

2. Now suppose that $AB = DE$. (The same result follows if $AC = DF$, *mutatis mutandis*.)

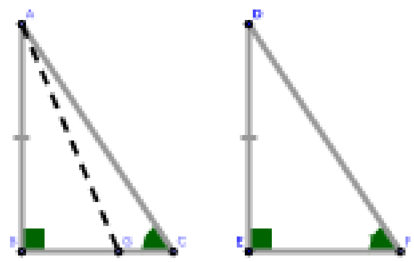

FIGURE 1.5.33. [1.26], case 2

If $BC \neq EF$, suppose that $EF = BG$. Join AG. Then the triangles $\triangle ABG$, $\triangle DEF$ have the two sides AB, BG of one respectively equal to the two sides DE, EF of the other where $\angle ABG = \angle DEF$ by hypothesis. By [1.4], $\angle AGB = \angle DFE$. We also have that $\angle DFE = \angle ACB$ by hypothesis. Hence, $\angle AGB = \angle ACB$ [Axiom 1.1]; that is, the exterior angle of the triangle $\triangle ACG$ is equal to an interior and non-adjacent angle, a contradiction [1.16]. Hence $BC = EF$.

Consequently, the triangles $\triangle ABC$, $\triangle DEF$ have the sides AB, BC of one respectively equal to the sides DE, EF of the other where interior angles $\angle ABC$ and $\angle DEF$ are equal. By [1.4], $\triangle ABC \cong \triangle DEF$. □

This proposition, together with [1.4] and [1.8], includes all the cases of the congruence of two triangles.

Case 1 may also be proved immediately by superposition: it is evident if $\triangle ABC$ is applied to $\triangle DEF$ such that the point B coincides with point E and the segment BC coincides with EF that, since $BC = EF$, point C coincides with point F. And since the angles $\angle ABC$, $\angle BCA$ are respectively equal to the angles $\angle DEF$, $\angle EFD$, the segments BA, CA will coincide with ED, FD. Hence, $\triangle ABC \cong \triangle DEF$.

Exercises.

1. The endpoints of the base of an isosceles triangle are equally distant from any point on the perpendicular segment from the vertical angle on the base.

2. If the line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

3. The locus of a point which is equally distant from two fixed lines is the pair of lines which bisect the angles made by the fixed lines.

4. In a given straight line, find a point such that the perpendiculars from it on two given lines are equal. State also the number of solutions.

5. Prove that if two right triangles have hypotenuses of equal length and an acute angle of one is equal to an acute angle of the other, then they are congruent.

6. Prove that if two right triangles have equal hypotenuses and that if a side of one is equal in length to a side of the other, then the triangles they are congruent. (Note: this proves the special case of Side-Side-Angle congruency for right triangles.)

The bisectors of the three internal angles of a triangle are concurrent. Their point of intersection is called the **incenter** of the triangle.

7. The bisectors of two external angles and the bisector of the third internal angle are concurrent.

8. Through a given point, construct a straight line such that perpendiculars on it from two given points on opposite sides are equal to each other.

9. Through a given point, construct a straight line intersecting two given lines which forms an isosceles triangle with them.

1.6. Propositions from Book I: 27-48

Additional definitions regarding parallel lines:

Parallel Lines. 38. If two straight lines in the same plane do not meet at any finite distance, they are said to be *parallel*. If rays or segments can be extended into lines which do not meet at any finite distance, they are also said to be *parallel*.

39. A *parallelogram* is a quadrilateral where both pairs of opposite sides are parallel.

40. The segment joining either pair of opposite angles of a quadrilateral is called a *diagonal*. See Fig. 1.6.1.

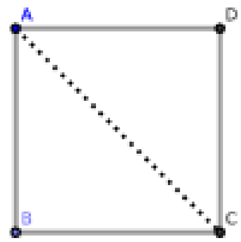

FIGURE 1.6.1. [Def 1.40] AC is a diagonal of the square $\boxdot ABCD$

41. A quadrilateral which has one pair of opposite sides parallel is called a *trapezium*.

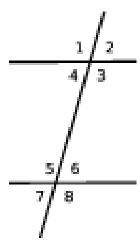

FIGURE 1.6.2. [Def. 1.42]

42. When a straight line intersects two other straight lines at two distinct points, between them are eight angles (see Fig. 1.6.2). Angles 1 and 2, 7 and 8 are called *exterior angles*; 3 and 4, 5 and 6 are called *interior angles*; 4 and 6, 3 and 5 are called *alternate angles*; 1 and 5, 2 and 6, 3 and 8, 4 and 7 are called *corresponding angles*. These definitions hold when we replace straight lines with either rays or segments, *mutatis mutandis*.

PROPOSITION 1.27. *PARALLEL LINES I. Suppose a straight line intersects two straight lines at one and only point point each. If the alternate angles are equal, then the lines are parallel.*

PROOF. If a straight line (EF) intersects two straight lines (AB, CD) such that the alternate angles are equal ($\angle AEF = \angle EFD$), then these lines are parallel ($AB \parallel CD$).

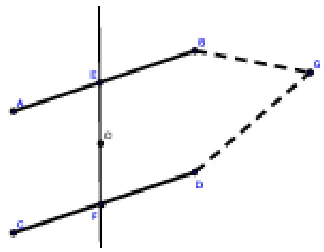

FIGURE 1.6.3. [1.27]

If $AB \nparallel CD$, then they must meet at some finite distance; suppose that they intersect at point G. Then $\triangle EGF$ is a triangle where $\angle AEF$ is an exterior angle and $\angle EFD$ a non-adjacent interior angle. By [1.16], $\angle AEF > \angle EFD$, which contradicts our hypothesis that $\angle AEF = \angle EFD$. Hence, $AB \parallel CD$. □

PROPOSITION 1.28. *1) PARALLEL LINES II. If a straight line intersecting two straight lines at one and only point point each makes the exterior angle equal to its corresponding interior angle, the two straight lines are parallel.*

2) PARALLEL LINES III. If a straight line intersecting two straight lines at one and only point point makes two interior angles on the same side equal to two right angles, the two straight lines are parallel.

PROOF. If a straight line (EF) intersects two straight lines (AB, CD) at one and only point point each such that the exterior angle ($\angle EGB$) equals its corresponding interior angle ($\angle GHD$), or if it makes two interior angles on the same side ($\angle BGH$, $\angle DHG$) equal to two right angles, then the two lines are parallel ($AB \parallel CD$).

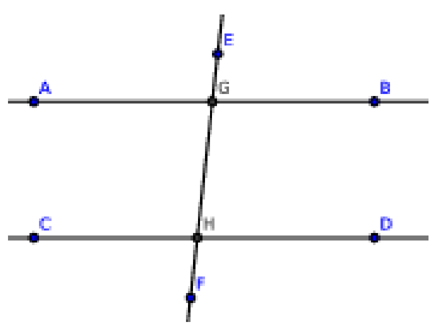

FIGURE 1.6.4. [1.28]

We prove each claim separately:

1. Suppose that $\angle EGB = \angle GHD$. Since the lines AB, EF intersect at G, $\angle AGH = \angle EGB$ [1.15]. It follows that $\angle AGH = \angle GHD$. Since these are alternate angles, by [1.27], $AB \parallel CD$.

2. Now suppose that the sum $\angle BGH + \angle DHG$ equals two right angles. Since $\angle AGH$ and $\angle BGH$ are adjacent angles, the sum $\angle AGH + \angle BGH$ equals two right angles [1.13]. If we remove the common angle $\angle BGH$, we have that $\angle AGH = \angle DHG$, and these are alternate angles. By [1.27], $AB \parallel CD$. □

PROPOSITION 1.29. *PARALLEL LINES IV. If a straight line intersects two parallel straight lines at one and only one point each, then:*

1) corresponding alternate angles are equal to each other,

2) exterior angles are equal to corresponding interior angles,

3) the sum of interior angles on the same side is equal to two right angles.

PROOF. If a straight line EF intersects two parallel straight lines AB, CD at one and only one point each, we claim that:

1. alternate angles $\angle AGH$, $\angle GHD$ are equal;

2. the exterior angle $\angle EGB$ equals its corresponding interior angles $\angle GHD$;

3. the sum of the two interior angles $\angle HGB + \angle GHD$ equals two right angles.

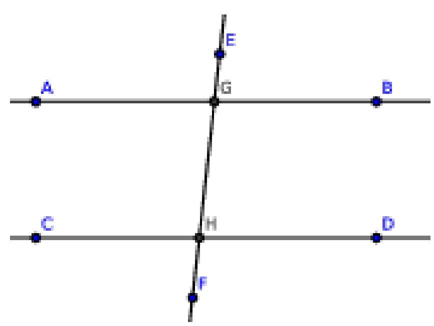

FIGURE 1.6.5. [1.29]

We prove each claim separately:

1. If $\angle AGH \neq \angle GHD$, one must be greater than the other. Wlog, suppose that $\angle AGH > \angle GHD$. Then we obtain the inequality

$$\angle AGH + \angle BGH > \angle GHD + \angle BGH$$

where $\angle AGH + \angle BGH$ is equal to the sum of two right angles [1.13]. It follows that the sum $\angle GHD + \angle BGH$ is less than two right angles. By [Axiom 1.12], the lines AB, CD meet at some finite distance, a contradiction – since they are parallel by hypothesis, they cannot meet at any finite distance. Hence, the angle $\angle AGH$ is not unequal to $\angle GHD$; or, $\angle AGH = \angle GHD$.

2. Since $\angle EGB = \angle AGH$ [1.15] and $\angle GHD = \angle AGH$ by part 1 of this proof, it follows that $\angle EGB = \angle GHD$ [Axiom 1.1].

3. Since $\angle AGH = \angle GHD$ by part 1 of this proof, we obtain

$$\angle AGH + \angle HGB = \angle GHD + \angle HGB$$

where the sum $\angle AGH + \angle BGH$ equals the sum of two right angles. It follows that the sum $\angle GHD + \angle HGB$ equals the sum of two right angles. □

COROLLARY. *1. EQUIVALENT STATEMENTS REGARDING PARALLEL LINES. Suppose two straight lines are intersected by a third straight line at one and only one point. The two straight lines are parallel if and only if any one of these three properties hold:*

1) corresponding alternate angles are equal;

2) exterior angles equal their corresponding interior angles;

3) the sum of the interior angles on the same side are equal to two right angles.

COROLLARY. *2. We may replace the straight lines in [1.29, Cor. 1] with segments or rays, mutatis mutandis.*

Exercises.

Note: We may use [1.31] (that we may construct a straight line parallel to any given straight line) in the proofs of these exercises since the proof of [1.31] does not employ [1.29].

1. Demonstrate both parts of [1.28] without using [1.27].

2. If $\angle ACD$, $\angle BCD$ are adjacent angles, any parallel to AB will meet the bisectors of these angles at points equally distant from where it meets CD.

4. If any other secant is constructed through the midpoint O of any straight line terminated by two parallel straight lines, the intercept on this line made by the parallels is bisected at O.

5. Two straight lines passing through a point equidistant from two parallels intercept equal segments on the parallels.

6. The perimeter of the parallelogram, formed by constructing parallels to two sides of an equilateral triangle from any point in the third side, is equal to twice the side.

7. If the opposite sides of a hexagon are equal and parallel, its diagonals are concurrent.

8. If two intersecting segments are respectively parallel to two others, the angle between the former is equal to the angle between the latter. For, if AB, AC are respectively parallel to DE, DF and if AC, DE intersect at G, the angles at points A, D are each equal to the angle at G [1.29].

PROPOSITION 1.30. *TRANSITIVITY OF PARALLEL LINES. Straight lines parallel to the same straight line are also parallel to one another.*

PROOF. Construct straight lines AB, CD, EF such that $AB \parallel EF$ and $CD \parallel EF$. We claim that $AB \parallel CD$.

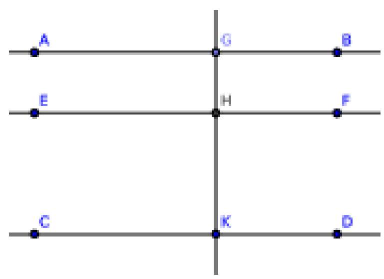

FIGURE 1.6.6. [1.30]

Construct any secant GHK. Since $AB \parallel EF$, the angle $\angle AGH = \angle GHF$ [1.29]. Similarly, the angle $\angle GHF = \angle HKD$ [1.29]. By [Axiom 1.1], $\angle AGK = \angle GKD$, and by [1.27], we have that $AB \parallel CD$. □

COROLLARY. *1. AB, CD, EF, and GK in [1.30] may be replaced by rays and/or straight-line segments, mutandis mutandis.*

PROPOSITION 1.31. *CONSTRUCTION OF A PARALLEL LINE. We wish to construct a straight line which is parallel to a given straight line and passes through a given point.*

PROOF. We wish to construct a straight line (CE) which is parallel to a given straight line (AB) and passes through a given point (C).

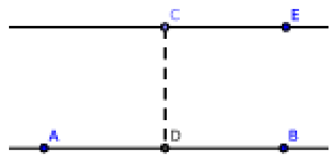

FIGURE 1.6.7. [1.31]

Take any point D on AB. Join CD [Postulate 1.1]. Choose any point E such that by joining points C and E we obtain $\angle DCE = \angle ADC$ [1.23]. By [1.29, Cor. 1], $AB \parallel CE$. □

COROLLARY. *1. AB and CE in [1.31] may be replaced by rays and/or straight-line segments, mutatis mutandis.*

Definition:

43. The *altitude* of a triangle is the perpendicular segment from the triangle's base to the base's opposing vertex.

Exercises.

1. Given the altitude of a triangle and the base angles, construct the triangle.

2. From a given point, construct a segment to a given segment such that the resultant angle is equal in measure to a given angle. Show that there are two solutions.

3. Prove the following construction for trisecting a given line AB:

On AB, construct an equilateral $\triangle ABC$. Bisect the angles at points A, B by the lines AD, BD. Through D, construct parallels to AC, BC, intersecting AB at E, F. Claim: E and F are the points of trisection of AB.

4. Inscribe a square in a given equilateral triangle such that its base stands on a given side of the triangle.

5. Through two given points on two parallel lines, construct two segments forming a lozenge with given parallels.

6. Between two lines given in position, place a segment of given length which is parallel to a given line. Show that there are two solutions.

PROPOSITION 1.32. *EXTERIOR ANGLES AND SUMS OF ANGLES IN A TRIANGLE. In any triangle, if one of the sides is extended, then:*

1) the exterior angle equals the sum of the its interior and opposite angles;

2) the sum of the three interior angles of the triangle equals two right angles.

PROOF. Construct $\triangle ABC$ and wlog extend side AB to segment BD. We claim that the external angle $\angle CBD$ equals the sum of the two internal non-adjacent angles ($\angle BAC + \angle ACB$) and that the sum of the three internal angles ($\angle BAC + \angle ACB + \angle ABC$) equals two right angles. Or: $\angle CBD = \angle BAC + \angle ACB$ and $\angle BAC + \angle ACB + \angle ABC =$two right angles.

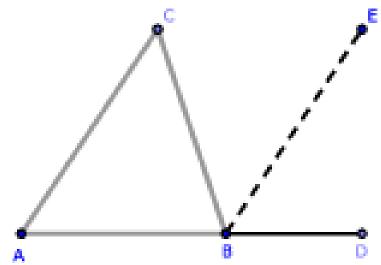

FIGURE 1.6.8. [1.32]

Construct $BE \parallel AC$ [1.31]. Since BC intersects the parallels BE and AC, we have that $\angle EBC = \angle ACB$ [1.29]. Also, since AB intersects the parallels BE and AC, we have that $\angle EBD = \angle BAC$ [1.29]. Since $\angle CBD = \angle EBC + \angle EBD$, we have that

$$\angle CBD = \angle ACB + \angle BAC$$

This proves the first claim.

Adding $\angle ABC$ to this equality, we obtain

$$\angle ABC + \angle CBD = \angle ABC + \angle ACB + \angle BAC$$

But the sum $\angle ABC + \angle CBD$ equals two right angles [1.13]. Hence, the sum $\angle ABC + \angle ACB + \angle BAC$ equals two right angles. □

COROLLARY. *1. If a right triangle is isosceles, then each base angle equals half of a right angle.*

COROLLARY. *2. If two triangles have two angles in one respectively equal to two angles in the other, then their remaining pair of angles is also equal.*

COROLLARY. *3. Since a quadrilateral can be divided into two triangles, the sum of its angles equals four right angles.*

COROLLARY. *4. If a figure of n sides is divided into triangles by drawing diagonals from any one of its angles, we will obtain $(n{-}2)$ triangles. Hence, the sum of its angles equals $2(n{-}2)$ right angles.*

COROLLARY. *5. If all the sides of any convex polygon are extended, then the sum of the external angles equals to four right angles.*

COROLLARY. *6. Each angle of an equilateral triangle equals two-thirds of a right angle.*

COROLLARY. *7. If one angle of a triangle equals the sum of the other two, then it is a right angle.*

COROLLARY. *8. Every right triangle can be divided into two isosceles triangles by a line constructed from the right angle to the hypotenuse.*

Exercises.

1. Trisect a right angle.

2. If the sides of a polygon of n sides are extended, then the sum of the angles between each alternate pair is equal to $2(n{-}4)$ right angles.

3. If the line which bisects the external vertical angle is parallel to the base, then the triangle is isosceles.

4. If two right triangles $\triangle ABC$, $\triangle ABD$ are on the same hypotenuse AB and if the vertices C and D are joined, then the pair of angles standing opposite any side of the resulting quadrilateral are equal.

5. The three altitudes of a triangle are concurrent. Note: We are proving the existence of the **orthocenter**[13] of a triangle: the point where the three altitudes intersect, and one of a triangle's **points of concurrency**[14]. (Hint: Solve using [1.34].)

6. The bisectors of two adjacent angles of a parallelogram are at right angles. (Hint: Solve using [1.34].)

7. The bisectors of the external angles of a quadrilateral form a circumscribed quadrilateral, the sum of whose opposite angles equals two right angles.

[13] http://mathworld.wolfram.com/Orthocenter.html
[14] http://www.mathopenref.com/concurrentpoints.html

8. If the three sides of one triangle are respectively perpendicular to those of another triangle, the triangles are equiangular. (This problem may be delayed until the end of chapter 1.)

9. Construct a right triangle being given the hypotenuse and the sum or difference of the sides.

10. The angles made with the base of an isosceles triangle by altitudes from its endpoints on the equal sides are each equal to half the vertical angle.

11. The angle included between the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to half the vertical angle.

12. In the construction of [1.18], prove that the angle $\angle DBC$ is equal to half the difference of the base angles.

13. If A, B, C denote the angles of a triangle, prove that $\frac{1}{2}(A+B)$, $\frac{1}{2}(B+C)$, and $\frac{1}{2}(A+C)$ are the angles of a triangle formed by any side, the bisectors of the external angles between that side, and the other extended sides.

PROPOSITION 1.33. *PARALLEL SEGMENTS. Segments which join adjacent endpoints of two equal, parallel segments are themselves parallel and equal in length.*

PROOF. If segments AC, BD join adjacent endpoints of two equal, parallel segments AB, CD, then $AC = BD$ and $AC \parallel BD$.

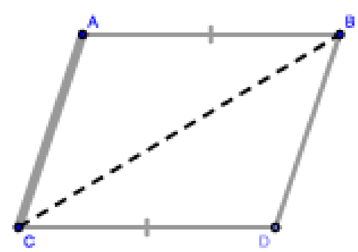

FIGURE 1.6.9. [1.33]

Join BC. Since $AB \parallel CD$ by hypothesis and BC intersects them, we have that $\angle ABC = \angle DCB$ [1.29]. Hence we have that $\triangle ABC$, $\triangle DCB$ have the sides AB, BC in one respectively equal to the sides DC, BC in the other where the interior angles $\angle ABC$, $\angle DCB$ are equal. By [1.4], $\triangle ABC \cong \triangle DCB$, and so $AC = BD$ and $\angle ACB = \angle CBD$. Since $\angle ACB$, $\angle CBD$ are alternate angles, by [1.27], $AC \parallel BD$. $\square$

COROLLARY. *1. [1.33] holds for straight lines and rays, mutatis mutandis.*

COROLLARY. *2. Figure $\boxdot ABDC$ is a parallelogram [Def. 1.39].*

Exercises.

1. Prove that if two segments AB, BC are respectively equal and parallel to two other segments DE, EF, then the segment AC joining the endpoints of the former pair is equal in length to the segment DF joining the endpoints of the latter pair.

PROPOSITION 1.34. *OPPOSITE SIDES AND OPPOSITE ANGLES OF PARALLELOGRAMS. The opposite sides and the opposite angles of a parallelogram are equal to one another and either diagonal bisects the parallelogram.*

PROOF. Construct $\boxdot ABCD$. We claim that $AB = CD$, $AC = BD$, $\angle CAB = \angle CDB$, and $\angle ACD = \angle ABD$. Furthermore, we claim that either diagonal (CB, AD) bisects the parallelogram.

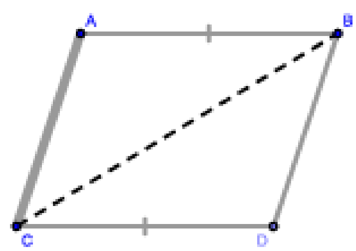

FIGURE 1.6.10. [1.34]

Join BC. Since $AB \parallel CD$ and BC intersects them, $\angle ABC = \angle DCB$ and $\angle ACB = \angle CBD$ [1.29]. Hence the triangles $\triangle ABC$, $\triangle DCB$ have the two angles $\angle ABC$, $\angle ACB$ in one respectively equal to the two angles $\angle BCD$, $\angle CBD$ in the other with side BC in common. By [1.26], $\triangle ABC \cong \triangle DCB$, and so $AB = CD$, $AC = BD$.

We also have that $\angle BAC = \angle BDC$, $\angle ACD = \angle ACB + \angle DCB$ and $\angle ABD = \angle CBD + \angle ABC$. By the equalities above, we obtain

$$\begin{aligned} \angle ACD &= \angle ACB + \angle DCB \\ &= \angle CBD + \angle ABC \\ &= \angle ABD \end{aligned}$$

or, that opposite angles are equal. Since $\boxdot ABCD = \triangle ABC \oplus \triangle DEF$ and $\triangle ABC \cong \triangle DCB$ (and hence the triangles have that same area), the diagonal bisects the parallelogram.

The remaining case follows *mutatis mutandis* if we join AD instead of BC.

□

COROLLARY. *1. The area of* $\boxdot ABDC$ *is double the area of* $\triangle ACB$ *(*$\triangle BCD$*).*

COROLLARY. *2. If one angle of a parallelogram is a right angle, each of its angles are right angles.*

COROLLARY. *3. If two adjacent sides of a parallelogram are equal in length, then it is a lozenge.*

COROLLARY. *4. If both pairs of opposite sides of a quadrilateral are equal in length, it is a parallelogram.*

COROLLARY. *5. If both pairs of opposite angles of a quadrilateral are equal, it is a parallelogram.*

COROLLARY. *6. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.*

COROLLARY. *7. If both diagonals of a quadrilateral bisect the quadrilateral, it is a parallelogram.*

COROLLARY. *8. If the adjacent sides of a parallelogram are equal, its diagonals bisect its angles.*

COROLLARY. *9. If the adjacent sides of a parallelogram are equal, its diagonals intersect at right angles.*

COROLLARY. *10. In a right parallelogram, the diagonals are equal in length.*

COROLLARY. *11. If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a lozenge.*

COROLLARY. *12. If a diagonal of a parallelogram bisects the angles whose vertices it joins, the parallelogram is a lozenge.*

Exercises.

1. Show that the diagonals of a parallelogram bisect each other.

2. If the diagonals of a parallelogram are equal, each of its angles are right angles.

3. Divide a segment into any number of equal parts.

4. The segments joining the adjacent endpoints of two unequal parallel segments will meet when extended on the side of the shorter parallel.

5. If two opposite sides of a quadrilateral are parallel but unequal in length and the other pair are equal but not parallel, then its opposite angles are supplemental.

6. Construct a triangle being given the midpoints of its three sides.

PROPOSITION 1.35. *AREAS OF PARALLELOGRAMS ON THE SAME BASE AND ON THE SAME PARALLELS. Parallelograms on the same base and between the same parallels are equal in area.*

PROOF. Parallelograms on the same base (BC) and between the same parallels (AF, BC) are equal in area. The proof follows in three cases.

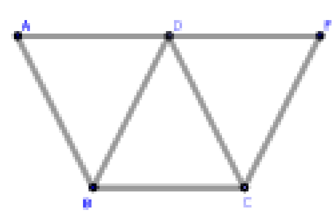

FIGURE 1.6.11. [1.35], case 1

1. Construct the parallelograms $\boxdot ADCB$, $\boxdot FDBC$ on the common base BC. Notice that side AD of $\boxdot ADCB$ and side DF of $\boxdot FDBC$ intersect only at point D. By [1.34], each parallelogram is double the area of the triangle $\triangle BCD$. Hence, $\boxdot ADCB = \boxdot FDBC$.

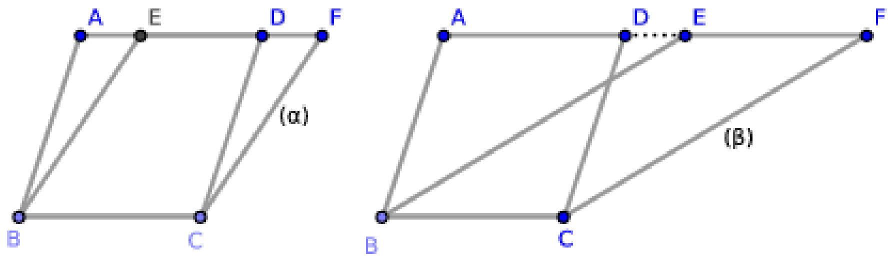

FIGURE 1.6.12. [1.35], cases 2 and 3

2. Construct the parallelograms $\boxdot ADCB$ and $\boxdot EFCB$ such that side AD of $\boxdot ADCB$ and side EF of $\boxdot EFCB$ intersect at more than one point (Fig. 1.6.12(α)). Because $\boxdot ABCD$ is a parallelogram, $AD = BC$ [1.34]; because $\boxdot BCEF$ is a parallelogram, $EF = BC$. Hence, $AD = EF$.

Removing ED, we have that the sides BA, AE in $\triangle BAE$ are respectively equal to the two sides CD, DF in $\triangle CDF$ and that $\angle BAE = \angle CDF$ [1.29, Cor. 1]. By [1.4], $\triangle BAE = \triangle CDF$. Notice that

$$AFCB = \boxdot EFCB + \triangle BAE = \boxdot ADCB + \triangle CDF$$

By the equality of the area of the triangles, it follows that $\boxdot ADCB = \boxdot EFCB$.

3. Construct the parallelograms $\boxdot ADCB$ and $\boxdot EFCB$ such that side AD of $\boxdot ADCB$ and side EF of $\boxdot EFCB$ do not intersect (Fig. 1.6.12(β)). As in case 2, we have that $AD = BC$ and $EF = BC$. Construct segment DE. Then we have that the sides BA, AE in $\triangle BAE$ are respectively equal to the two sides CD, DF in $\triangle CDF$ and that $\angle BAE = \angle CDF$ [1.29]. By [1.4], $\triangle BAE = \triangle CDF$. As in case 2, it follows that $\boxdot ADCB = \boxdot EFCB$. $\square$

Alternatively:

PROOF. Construct the triangles $\triangle BAE$, $\triangle CDF$ as well as the segment DE if necessary to create the quadrilateral $AFCB$. Notice that $\triangle BAE$, $\triangle CDF$ have the sides AB, BE in one respectively equal to the sides DC, CF in the other [1.34] and that $\angle BAE = \angle DCF$ [1.29, #8]. Hence, $\triangle BAE \cong \triangle CDF$. Since

$$AFCB = \boxdot EFCB + \triangle BAE = \boxdot ADCB + \triangle CDF$$

the proof follows. $\square$

PROPOSITION 1.36. *AREAS OF PARALLELOGRAMS ON EQUAL BASES AND ON THE SAME PARALLELS. Parallelograms on equal bases and on the same parallels are equal in area.*

PROOF. Parallelograms ($\boxdot ADCB$, $\boxdot EHGF$) on equal bases (BC, FG) and standing between the same parallels (AH, BG) are equal in area.

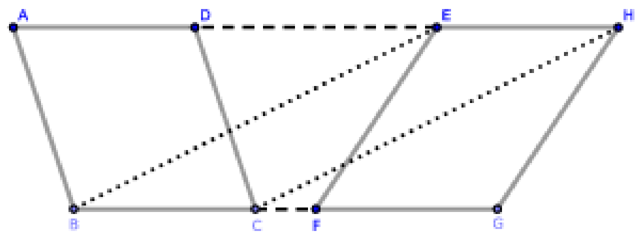

FIGURE 1.6.13. [1.36]

Join DE, CF, BE, CH. Since $\boxdot EHGF$ is a parallelogram, $FG = EH$ [1.34]. Since $BC = FG$ by hypothesis, we have that $BC = EH$ [Axiom 1.1]. Since BE, CH are also parallel and join the adjacent endpoints of EH, BC, $\boxdot EBCH$ is a parallelogram.

Again, since the parallelograms $\boxdot ADCB$, $\boxdot EBCH$ stand on the same base BC and between the same parallels BC, AH, $\boxdot ADCB = \boxdot EBCH$ [1.35]. Similarly, $\boxdot EBCH = \boxdot EHGF$. By [Axiom 1.1], $\boxdot ADCB = \boxdot EBCH$. □

Exercise.

1. Prove this proposition without joining BE, CH.

PROPOSITION 1.37. *TRIANGLES OF EQUAL AREA I. Triangles which stand on the same base and in the same parallels are equal in area.*

PROOF. We claim that triangles ($\triangle ABC$, $\triangle DBC$) on the same base (BC) and standing between the same parallels (AD, BC) are equal in area.

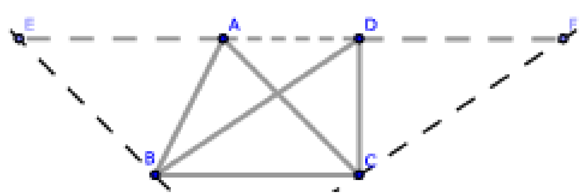

FIGURE 1.6.14. [1.37]

Construct $BE \parallel AC$ and $CF \parallel BD$ [1.31] and extend AD to segments AE and DF. It follows that the figures $\boxdot AEBC$, $\boxdot DBCF$ are parallelograms.

Since they stand on the same base BC and between the same parallels BC, EF, $\boxdot AEBC = \boxdot DBCF$ [1.35].

Notice that area of the triangle $\frac{1}{2} \cdot \triangle ABC = \boxdot AEBC$ because the diagonal AB bisects $\boxdot AEBC$ [1.34]. Similarly, $\frac{1}{2} \cdot \triangle DBC = \boxdot DBCF$. Since halves of equal magnitudes are equal [Axiom 1.7], we have that $\triangle ABC = \triangle DBC$. □

Exercises.

1. If two triangles of equal area stand on the same base but on opposite sides, the segment joining their vertices is bisected by the base.

2. Construct a triangle equal in area to a given quadrilateral figure.

3. Construct a triangle equal in area to a given polygon.

4. Construct a lozenge equal in area to a given parallelogram and having a given side of the parallelogram for base.

5. Given the base and the area of a triangle, find the locus of the vertex.

PROPOSITION 1.38. *TRIANGLES OF EQUAL AREA II. Triangles which stand on equal bases and in the same parallels are equal in area.*

PROOF. By a construction analogous to [1.37], we have that the triangles are the halves of parallelograms, standing on equal bases and between the same parallels. Hence, they are the halves of equal parallelograms [1.36] and so are equal in area to each other. □

Exercises.

1. Every median of a triangle bisects the triangle.

2. If two triangles have two sides of one respectively equal to two sides of the other and where the interior angles are supplemental, their areas are equal.

3. If the base of a triangle is divided into any number of equal segments, then segments constructed from the vertex to the points of division divide the whole triangle into as many equal parts.

4. The diagonal of a parallelogram and segments from any point on the diagonal to the angular points through which the diagonal does not pass divide the parallelogram into four triangles which are equal (in a two by two fashion).

5. One diagonal of a quadrilateral bisects the other if and only if it also bisects the quadrilateral.

6. If two triangles $\triangle ABC$, $\triangle ABD$ stand on the same base AB and between the same parallels, and if a parallel to AB meets the sides AC, BC at the points E, F as well as the sides AD, BD at the points G, H, then $EF = GH$.

7. If instead of triangles on the same base we have triangles on equal bases and between the same parallels, the intercepts made by the sides of the triangles on any parallel to the bases are equal in length.

8. If the midpoints of any two sides of a triangle are joined, the triangle formed with the two half sides has an area equal to one-fourth of the whole.

9. The triangle whose vertices are the midpoints of two sides and any point in the base of another triangle has an area equal to one-fourth the area of that triangle.

10. Bisect a given triangle by a segment constructed from a given point in one of the sides.

11. Trisect a given triangle by three segments constructed from a given point within it.

12. Prove that any segment through the intersection of the diagonals of a parallelogram bisects the parallelogram.

13. The triangle formed by joining the midpoint of one of the non-parallel sides of a trapezium to the endpoints of the opposite side is equal in area to half the area of the trapezium.

PROPOSITION 1.39. *TRIANGLES OF EQUAL AREA III. Triangles which are equal in area and stand on the same base and on the same side of the base also stand on the same parallels.*

PROOF. Equal triangles ($\triangle BAC$, $\triangle BDC$) on the same base (BC) and on the same side of the base also stand between the same parallels (AD, BC).

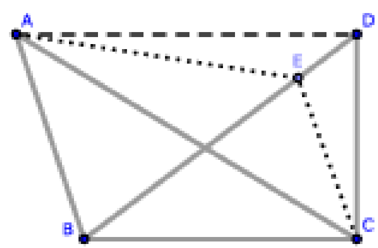

FIGURE 1.6.15. [1.39]

Join AD. If $AD \nparallel BC$, suppose that $AE \parallel BC$ where the segments intersect at point E. Join EC. Since the triangles $\triangle BEC$, $\triangle BAC$ stand on the same base BC and between the same parallels BC, AE, we have that $\triangle BEC = \triangle BAC$ [1.37]. By hypothesis, $\triangle BAC = \triangle BDC$. Therefore, $\triangle BEC = \triangle BDC$ [Axiom 1.1]. But $\triangle BDC = \triangle BEC + \triangle EDC$, a contradiction. Hence, we must have that $AD \parallel BC$. $\square$

PROPOSITION 1.40. *TRIANGLES OF EQUAL AREA IV. Triangles which are equal in area and stand on equal bases and on the same side of their bases stand on the same parallels.*

PROOF. Triangles which are equal in area ($\triangle ABC$, $\triangle DEF$) as well as stand on equal bases (BC, EF) and on the same side of their bases stand on the same parallels.

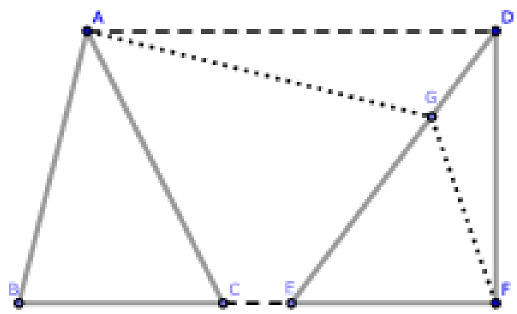

FIGURE 1.6.16. [1.40]

Join AD. If $AD \nparallel BF$, let $AG \parallel BF$. Join GF. Since the triangles $\triangle GEF$, $\triangle ABC$ stand on equal bases BC, EF and between the same parallels BF, AG, we have that $\triangle GEF = \triangle ABC$ [1.38]; but $\triangle DEF = \triangle ABC$ by hypothesis. Hence $\triangle GEF = \triangle DEF$ [Axiom 1.1]. However, $\triangle DEF = \triangle GEF + \triangle DGF$, a contradiction. Therefore, we must have that $AD \parallel BF$. □

Exercises.

1. Triangles with equal bases and altitudes are equal in area.

2. The segment joining the midpoints of two sides of a triangle is parallel to the third because the medians from the endpoints of the base to these points will each bisect the original triangle. Hence, the two triangles whose base is the third side and whose vertices are the points of bisection are equal in area.

3. The parallel to any side of a triangle through the midpoint of another bisects the third.

4. The segments which connect the midpoints of the sides of a triangle divide it into four congruent triangles.

5. The segment which connects the midpoints of two sides of a triangle is equal in length to half the third side.

6. The midpoints of the four sides of a convex quadrilateral, taken in order, are the angular points of a parallelogram whose area is equal to half the area of the quadrilateral.

7. The sum of the two parallel sides of a trapezium is double the length of the segment joining the midpoints of the two remaining sides.

8. The parallelogram formed by the segment which connects the midpoints of two sides of a triangle and any pair of parallels constructed through the same points to meet the third side is equal in area to half the area of the triangle.

9. The segment joining the midpoints of opposite sides of a quadrilateral and the segment joining the midpoints of its diagonals are concurrent.

PROPOSITION 1.41. *PARALLELOGRAMS AND TRIANGLES. If a parallelogram and a triangle stand on the same base and between the same parallels, then the parallelogram is double the area of the triangle.*

PROOF. If a parallelogram ($\boxdot ABCD$) and a triangle ($\triangle EBC$) stand on the same base (BC) and between the same parallels (AE, BC), then the parallelogram is double the area of the triangle.

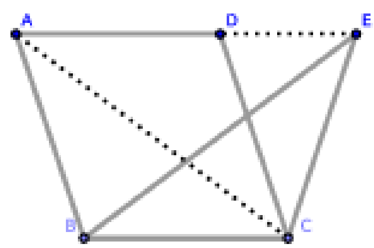

FIGURE 1.6.17. [1.41]

Join AC. By [1.34], $\boxdot ABCD = 2 \cdot \triangle ABC$, and by [1.37], $\triangle ABC = \triangle EBC$. Therefore, $\boxdot ABCD = 2 \cdot \triangle EBC$. □

COROLLARY. *1. If a triangle and a parallelogram have equal altitudes and if the base of the triangle is double of the base of the parallelogram, their areas are equal.*

COROLLARY. *2. Suppose we have two triangles whose bases are two opposite sides of a parallelogram and which have any point between these sides as a common vertex. Then the sum of the areas of these triangles equals half the area of the parallelogram.*

PROPOSITION 1.42. *CONSTRUCTION OF PARALLELOGRAMS I. Given an arbitrary triangle and an arbitrary acute angle, it is possible to construct a parallelogram equal in area to the triangle which also contains the given angle.*

PROOF. We wish to construct a parallelogram equal in area to a given triangle ($\triangle ABC$) which contains an angle equal to a given angle ($\angle RDS$).

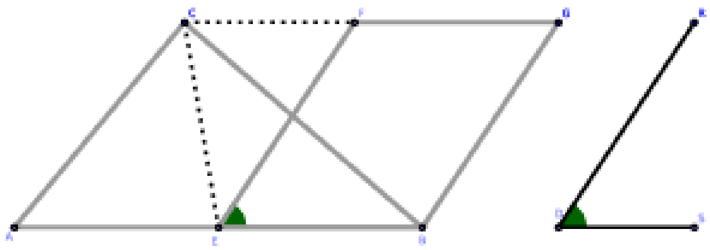

FIGURE 1.6.18. [1.42]

Bisect AB at E and join EC. Construct $\angle BEF = \angle RDS$ [1.23], $CG \parallel AB$, and $BG \parallel EF$ [1.31]. We claim that $\boxdot FEBG$ is the required parallelogram.

Because $AE = EB$ by construction, we have that $\triangle AEC = \triangle EBC$ [1.38]. Therefore, $\triangle ABC = 2 \cdot \triangle EBC$. We also have that $\boxdot FEBG = 2 \cdot \triangle EBC$ because each stands on the same base EB and between the same parallels EB and CG [1.41]. Therefore, $\boxdot FEBG = \triangle ABC$, and by construction, $\angle BEF = \angle RDS$. □

PROPOSITION 1.43. *COMPLEMENTARY AREAS OF PARALLELOGRAMS. Parallel segments through any point in one of the diagonals of a parallelogram divides the parallelogram into four smaller parallelograms: the two through which the diagonal does not pass are called the complements of the other two, and these complements are equal in area.*

PROOF. We claim that segments which are parallel to the sides of a parallelogram $\boxdot ABCD$ (specifically EF,GH) and pass through any point (K) on one of the diagonals (AC) of $\boxdot ABCD$ divide$\boxdot ABCD$ into four smaller parallelograms: the two through which the diagonal does not pass ($\boxdot EBGK$, $\boxdot HKFD$) are called the complements of the other two. We also claim that $\boxdot EBGK = \boxdot HKFD$.

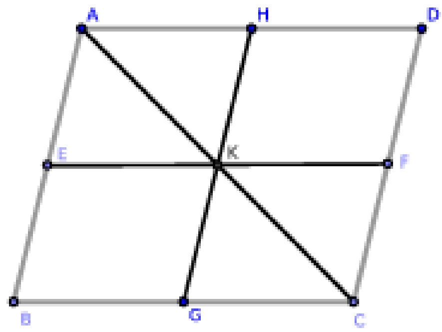

FIGURE 1.6.19. [1.43]

Because AC bisects the parallelograms $\boxdot ABCD$, $\boxdot AEKH$, $\boxdot KGCF$, we have that $\triangle ADC = \triangle ABC$, $\triangle AHK = \triangle AEK$, and $\triangle KFC = \triangle KGC$ [1.34]. Hence, we have that

$$\begin{array}{rcl} \boxdot EBGK & = & \triangle ABC - \triangle AEK - \triangle KGC \\ & & \shortparallel \\ \boxdot HKFD & = & \triangle ADC - \triangle AHK - \triangle KFC \end{array}$$

or simply $\boxdot HKFD = \boxdot EBGK$. □

COROLLARY. *1. If through some point K within parallelogram $\boxdot ABCD$ we have constructed parallel segments to its sides in order to make the parallelograms $\boxdot HKFD$, $\boxdot EBGK$ equal in area, then K is a point a the diagonal AC.*

COROLLARY. *2. $\boxdot ABGH = \boxdot AEFD$ and $\boxdot EBCF = \boxdot HGCD$.*

PROPOSITION 1.44. *CONSTRUCTION OF PARALLELOGRAMS II. Given an arbitrary triangle, an arbitrary angle, and an arbitrary segment, it is possible to construct a parallelogram equal in area to the triangle which contains the given angle and has a side equal in length to the given segment.*

PROOF. On a given segment (AB), we wish to construct a parallelogram equal in area to a given triangle ($\triangle NPQ$) which contains an equal to a given angle ($\angle RST$).

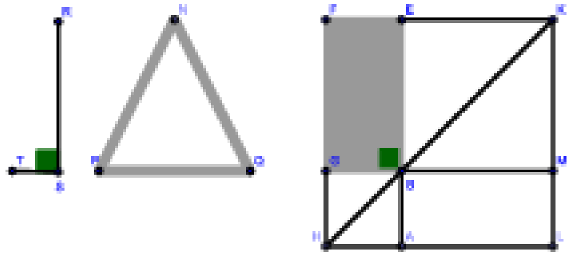

FIGURE 1.6.20. [1.44]

Construct the parallelogram $\boxdot BEFG$ such that $\boxdot BEFG = \triangle NPQ$ [1.42] where $\angle GBE = \angle RST$ and where AB and BE form the segment AE. Also construct segment $AH \parallel BG$ [1.31]. Extend FG to intersect AH at point H. Join HB.

Because $HA \parallel FE$ and HF intersects them, the sum $\angle AHF + \angle HFE$ equals two right angles [1.29]. It follows that the sum $\angle BHF + \angle HFE$ is less than two right angles since $\angle AHF = \angle BHF + \angle BHA$. By [Axiom 1.12], if we extend segments HB and FE, they will intersect at some point K. Through K, construct $KL \parallel AB$ [1.31] and extend HA and GB to intersect KL at points L and M, respectively. We claim that $\boxdot BALM$ is a parallelogram which fulfills the required conditions.

By [1.43], $\boxdot BALM = \boxdot FGBE$. Recall that $\boxdot FGBE = \triangle NPQ$ by construction, and therefore $\boxdot BALM = \triangle NPQ$. Again, $\angle ABM = \angle EBG$ [1.15], and $\angle EBG = \angle RST$ by construction. Therefore, $\angle ABM = \angle RST$. Finally, $\boxdot BALM$ is constructed on the given segment AB. □

PROPOSITION 1.45. *CONSTRUCTION OF PARALLELOGRAMS III. Given an arbitrary angle and an arbitrary polygon, it is possible to construct a parallelogram equal in area to the given polygon which contains an angle equal to the given angle.*

PROOF. We wish to construct a parallelogram equal in area to a given polygon ($ABCD$) which contains an angle equal to a given angle ($\angle LMN$).

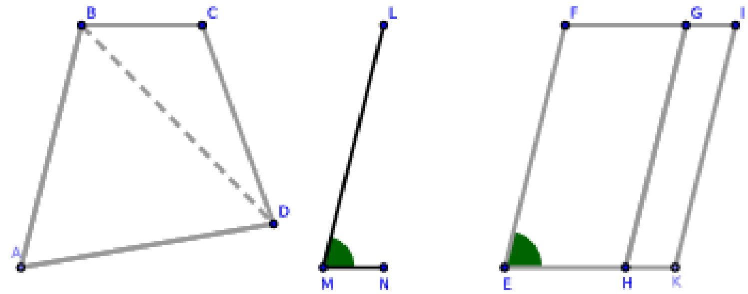

FIGURE 1.6.21. [1.45]

Join BD. Construct a parallelogram $\boxdot FEHG$ equal in area to the triangle $\triangle ABD$ where $\angle FEH = \angle LMN$ [1.42]. On segment GH, construct the parallelogram $\boxdot GHKI$ such that $\boxdot GHKI = \triangle BCD$ where $\angle GHK = \angle LMN$ [1.44]. We may continue to this algorithm for any additional triangles that remain in $ABCD$. Upon exhaustion of this process, we claim that $\boxdot FEKI$ is a parallelogram which fulfills the required conditions.

Because $\angle LMN = \angle FEH$ and $\angle GHK = \angle LMN$ by construction, we have that $\angle GHK = \angle FEH$. From this, we obtain

$$\angle GHK + \angle GHE \quad = \quad \angle FEH + \angle GHE$$

But since $HG \parallel EF$ and EH intersects them, the sum $\angle FEH + \angle GHE$ equals two right angles [1.29]. Hence, the sum of $\angle GHK + \angle GHE$ equals two right angles, and EH, HK form the segment EK [1.14, Cor. 1].

Similarly, because GH intersects the parallels FG, EK, the alternate angles $\angle FGH$, $\angle GHK$ are equal [1.29]. From this, we obtain

$$\angle FGH + \angle HGI \quad = \quad \angle GHK + \angle HGI$$

Since $GI \parallel HK$ and GH intersects them, the sum $\angle GHK + \angle HGI$ equals two right angles [1.29]. Hence, the sum $\angle FGH, + \angle HGI$ equals two right angles, and FG and GI form the segment FI [1.14, Cor. 1].

Again, because $\boxdot FEHG$ and $\boxdot GHKI$ are parallelograms, EF and KI are each parallel to GH. By [1.30], we have that $EF \parallel KI$ and $EK \parallel FI$. Therefore, $\boxdot FEKI$ is a parallelogram. And because the parallelogram $\boxdot FEHG = \triangle ABD$ by construction and $\boxdot GHKI = \triangle BCD$, the parallelogram $\boxdot FEKI = ABCD$. Since $\angle FEH = \angle LMN$, the proof follows. □

Exercises.

1. Construct a rectangle equal to the sum of $2, 3, ..., n$ number of polygons.

2. Construct a rectangle equal in area to the difference in areas of two given figures.

PROPOSITION 1.46. *CONSTRUCTION OF A SQUARE I. Given an arbitrary segment, it is possible to construct a square on that segment.*

PROOF. We wish to construct a square on a given segment (AB).

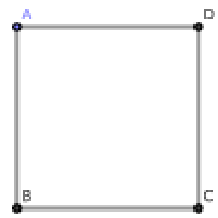

FIGURE 1.6.22. [1.46]

Construct $AD \perp AB$ [1.11] such that $AD = AB$ [1.3]. Through point D, construct $DC \parallel AB$ [1.31], and through point B construct $BC \parallel AD$ where BC and DC intersect only at point C. We claim that $\boxdot ABCD$ is the required square.

Because $\boxdot ABCD$ is a parallelogram, $AB = CD$ [1.34], and $AB = AD$ by construction. Therefore, $AD = CD$ and $AD = BC$ [1.34] and the four sides of $\boxdot ABCD$ are equal. It follows that $\boxdot ABCD$ is a lozenge and $\angle DAB$ is a right angle. Therefore, AC is a square [Def. 1.30]. □

Exercises.

1. Two squares have equal side-lengths if and only if the squares are equal in area.

2. The parallelograms about the diagonal of a square are squares.

3. If on the four sides of a square, or on the sides which are extended, points are taken equidistant from the four angles, they will be the angular points of another square, and similarly for a regular pentagon, hexagon, etc.

4. Divide a given square into five equal parts: specifically, four right triangles and a square.

PROPOSITION 1.47. *THE PYTHAGOREAN THEOREM. In a right triangle, the square of the length of the side opposite the right angle (the hypotenuse) is equal to the sum of the squares of the remaining side-lengths.*

PROOF. In a right triangle ($\triangle ABC$), we claim that the square on the hypotenuse (AB) is equal to the sum of the squares on the other two sides (AC, BC).

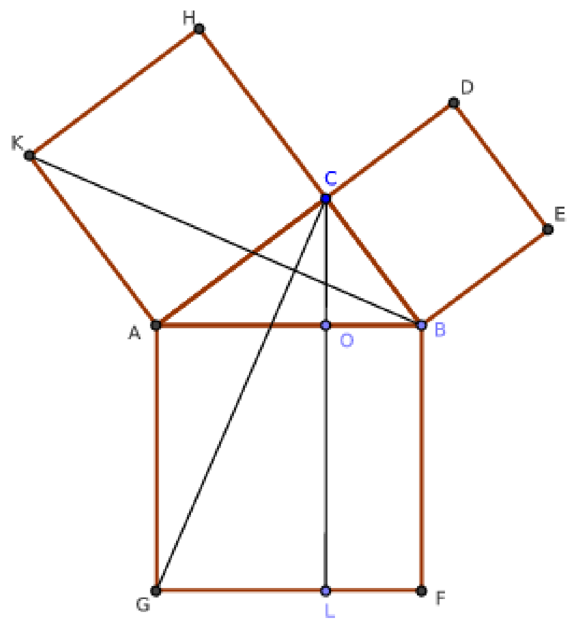

FIGURE 1.6.23. [1.47]

On the sides AB, BC, CA of $\triangle ABC$, construct squares [1.46]. Construct segment $CL \parallel AG$. Join CG, BK. Because $\angle ACB$ is right by hypothesis and $\angle ACH$ is right by construction, the sum $\angle ACB + \angle ACH$ equals two right angles. Therefore BC, CH form the segment BH [1.14]. Similarly, AC, CD for the segment AD.

Because $\angle BAG$, $\angle CAK$ are angles within a square, they are right angles. Hence, $\angle BAG = \angle CAK$; to each, add $\angle BAC$, and we obtain $\angle CAG = \angle KAB$.

Again, since $\boxdot BAGF$ and $\boxdot CHKA$ are squares, $BA = AG$, and $CA = AK$. Hence, the two triangles $\triangle CAG$, $\triangle KAB$ have the sides CA, AG in one respectively equal to the sides KA, AB in the other such that their interior angles are equal ($\angle CAG = \angle KAB$). By [1.4], $\triangle CAG \cong \triangle KAB$. But $\boxdot AGLO = 2 \cdot \triangle CAG$ because they are on the same base AG and between the same parallels (AG and CL), [1.41]. Similarly, the parallelogram $\boxdot CHKA = 2 \cdot \triangle KAB$ because they stand on the same base AK and between the same parallels (AK and BH). Since doubles of equal magnitudes are equal [Axiom 1.6], the parallelogram $\boxdot AGLO = \boxdot KACH$. Similarly, it can be shown that the parallelogram

$\boxdot OLFB = \boxdot DCBE$. Hence,

$$\boxdot AGFB = \boxdot AGLO \oplus \boxdot OLFB = \boxdot KACH + \boxdot DCBE$$

. $\square$

Alternatively:

PROOF. Construct the squares as in Fig. 1.6.24.

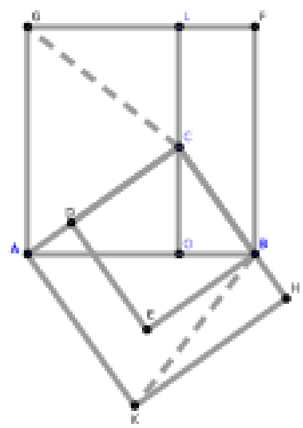

FIGURE 1.6.24. [1.47], alternate proof

Join CG, BK, and through C construct $OL \parallel AG$. Notice that $\angle GAK = \angle GAC + \angle BAC + \angle BAK$ and that $\angle BAG$, $\angle CAK$ are right angles. Removing $\angle BAC$ from each side of the equality, it follows that $\angle CAG = \angle BAK$.

Hence the triangles $\triangle CAG$, $\triangle BAK$ have the sides $CA = AK$, $AG = AB$, and $\angle CAG = \angle BAK$; by [1.4], $\triangle CAG \cong \triangle BAK$. By [1.41], since the parallelograms $\boxdot GAOL$, $\boxdot AKHC$ are respectively the doubles in area of the these triangles, we have that $\boxdot GAOL = \boxdot AKHC$. Similarly, $\boxdot LOBF = \boxdot DEBC$. Hence, . $\square$

The alternative proof is shorter since it is not necessary to prove that AC, CD are in one segment. Similarly, the proposition may be proved by taking any of the eight figures formed by turning the squares in all possible directions. Another simplification of the proof can be obtained by considering that the point A is such that one of the triangles $\triangle CAG$, $\triangle BAK$ can be turned round it in its own plane until it coincides with the other; hence, they are congruent.

Exercises.

1. The square on AC is equal in area to the rectangle $\boxdot AB.AO$, and the square on $\boxdot BC = \boxdot AB.BO$. (Note: $\boxdot AB.AO$ denotes that rectangle formed by the segments AB and AO as well as the area of that rectangle.)

2. The square on $\boxdot CO = \boxdot AO.OB$.

3. Prove that $AC^2 - BC^2 = AO^2 - BO^2$

4. Find a segment whose square is equal to the sum of two given squares.

5. Given the base of a triangle and the difference of the squares of its sides, the locus of its vertex is a segment perpendicular to the base.

6. The transverse segments BK, CG are perpendicular to each other.

7. If EG is joined, then $EG^2 = AC^2 + 4BC^2$.

8. The square constructed on the sum of the sides of a right triangle exceeds the square on the hypotenuse by four times the area of the triangle (see [1.46], Fig. 1.6.20, #3). More generally, if the vertical angle of a triangle is equal to the angle of a regular polygon of n sides, then the regular polygon of n sides, constructed on a segment equal to the sum of its sides exceeds the area of the regular polygon of n sides constructed on the base by n times the area of the triangle.

9. If AC and BK intersect at P and a segment is constructed through P which is parallel to BC, meeting AB at Q, then $CP = PQ$.

10. Each of the triangles $\triangle AGK$ and $\triangle BEF$ formed by joining adjacent corners of the squares is equal in area to the right triangle $\triangle ABC$. (Hint: use trigonometry.)

11. Find a segment whose square is equal to the difference of the squares on two segments.

12. The square on the difference of the sides AC, CB is less than the square on the hypotenuse by four times the area of the triangle.

13. If AE is joined, the segments AE, BK, CL, are concurrent.

14. In an equilateral triangle, three times the square on any side is equal to four times the square on the perpendicular to it from the opposite vertex.

15. We construct the square $\boxdot BEFG$ on BE, a part of the side BC of a square $\boxdot ABCD$, having its side BG in the continuation of AB. Divide the figure $AGFECD$ into three parts which will form a square.

16. Four times the sum of the squares on the medians which bisect the sides of a right triangle is equal to five times the square on the hypotenuse.

17. If perpendiculars fall on the sides of a polygon from any point and if we divide each side into two segments, then the sum of the squares on one set of alternate segments is equal to the sum of the squares on the remaining set.

18. The sum of the squares on segments constructed from any point to one pair of opposite angles of a rectangle is equal to the sum of the squares on the segments from the same point to the remaining pair.

19. Divide the hypotenuse of a right triangle into two parts such that the difference between their squares equals the square on one of the sides.

20. From the endpoints of the base of a triangle, let altitudes fall on the opposite sides. Prove that the sum of the rectangles contained by the sides and their lower segments is equal to the square on the base.

PROPOSITION 1.48. *THE CONVERSE OF THE PYTHAGOREAN THEOREM. If the square on one side of a triangle is equal to the sum of the squares on the remaining sides, then the angle opposite to the longest side (the hypotenuse) is a right angle.*

PROOF. If the square on one side (AB) of a triangle ($\triangle ABC$) equals the sum of the squares on the remaining sides (AC, CB), then the angle ($\angle ACB$) opposite to that side is a right angle. Or, if $AB^2 = AC^2 + BC^2$, then $\angle ACB$ is a right angle.

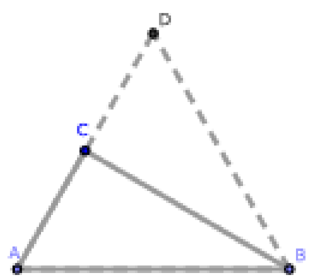

FIGURE 1.6.25. [1.48]

Construct $CD \perp CB$ [1.11] such that $CD = CA$ [1.3]. Join BD. Because $AC = CD$, $AC^2 = CD^2$. From this, we obtain

$$AC^2 + CB^2 = CD^2 + CB^2$$

But $AC^2 + CB^2 = AB^2$ by hypothesis, and $CD^2 + CB^2 = BD^2$ [1.46]. It follows that $AB^2 = BD^2$; hence $AB = BD$ [1.46, #1].

Again, because $AC = CD$ by construction and CB is a common side to the triangles $\triangle ACB$, $\triangle DCB$, we have that $AB = DB$ and $\angle ACB = \angle DCB$. But $\angle DCB$ is a right angle by construction, and so $\angle ACB$ is a right angle. $\square$

The above proof forms an exception to Euclid's demonstrations of converse propositions. The following is an indirect proof:

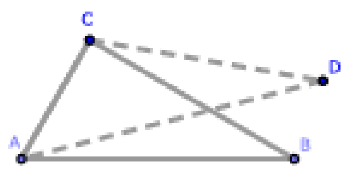

FIGURE 1.6.26. [1.48], alternative proof

PROOF. If $CB \not\perp AC$, construct $CD \perp CB$ such that $CD = CB$. Join AD. Then, as before, it can be shown that $AD = AB$. This is contrary to [1.7]. Hence, $\angle ACB$ is a right angle. □

Examination questions on chapter 1.

1. What is geometry?

2. What is geometric magnitude?

3. Name the primary concepts of geometry. (Ans. Points, lines, surfaces, and solids.)

4. What kinds of lines exist in geometry (Ans. Straight and curved.)

5. How is a straight line generated? (Ans. By connecting any three collinear points.)

6. How is a curved line generated? (Ans. By connecting any three non-collinear points.)

7. How may surfaces be divided? (Ans. Into planes and curved surfaces.)

8. How may a plane surface be generated?

9. Why has a point no dimensions?

10. Does a line have either width nor thickness?

11. How many dimensions does a surface possess?

12. What is plane geometry?

13. What portion of plane geometry forms the subject of this chapter?

14. What is the subject-matter of the remaining chapters?

15. How is a proposition proved indirectly?

16. What is meant by the inverse of a proposition?

17. What proposition is an instance of the Rule of Symmetry?

18. What are congruent figures?

19. What is another way to describe congruent figures? (Ans. They are said to be identically equal.)

20. Mention all the instances of equality which are not congruence that occur in chapter 1.

21. What is the difference between the symbols denoting congruence and identity?

22. Define adjacent, exterior, interior, and alternate angles, respectively.

23. What is meant by the projection of one line on another?

24. What are meant by the medians of a triangle?

25. What is meant by the third diagonal of a quadrilateral?

26. Mention some propositions in chapter 1 which are particular cases of more general ones that follow.

27. What is the sum of all the exterior angles of any polygon equal to?

28. How many conditions must be given in order to construct a triangle? (Ans. Three; such as the three sides, or two sides and an angle, etc.)

Chapter 1 exercises.

1. Any triangle is equal to a fourth part of the area which is formed by constructing through each vertex a line which is parallel to its opposite side.

2. The three altitudes of the first triangle in #1 are the altitudes at the midpoints of the sides of the second triangle.

3. Through a given point, construct a line so that the portion intercepted by the segments of a given angle are bisected at the point.

4. The three medians of a triangle are concurrent. (Note: we are proving the existence of the **centroid** of a triangle.)

5. Construct a triangle given two sides and the median of the third side.

6. In every triangle, the sum of the medians is less than the perimeter but greater than three-fourths of the perimeter.

7. Construct a triangle given a side and the two medians of the remaining sides.

8. Construct a triangle given the three medians.

9. The angle included between the perpendicular from the vertical angle of a triangle on the base and the bisector of the vertical angle is equal to half the difference of the base angles.

10. Find in two parallels two points which are equidistant from a given point and whose connecting line is parallel to a given line.

11. Construct a parallelogram given two diagonals and a side.

12. The shortest median of a triangle corresponds to the largest side.

13. Find in two parallels two points standing opposite a right angle at a given point and which are equally distant from it.

14. The sum of the distances of any point in the base of an isosceles triangle from the equal sides is equal to the distance of either endpoint of the base from the opposite side.

15. The three perpendiculars at the midpoints of the sides of a triangle are concurrent. Hence, prove that perpendiculars from the vertices on the opposite sides are concurrent [see #2].

16. Inscribe a lozenge in a triangle having for an angle one angle of the triangle.

17. Inscribe a square in a triangle having its base on a side of the triangle.

18. Find the locus of a point, the sum or the difference of whose distance from two fixed lines is equal to a given length.

19. The sum of the perpendiculars from any point in the interior of an equilateral triangle is equal to the perpendicular from any vertex on the opposite side.

20. Find a point in one of the sides of a triangle such that the sum of the intercepts made by the other sides on parallels constructed from the same point to these sides are equal to a given length.

21. If two angles exist such that their segments are respectively parallel, then their bisectors are either parallel or perpendicular.

22. Inscribe in a given triangle a parallelogram whose diagonals intersect at a given point.

23. Construct a quadrilateral where the four sides and the position of the midpoints of two opposite sides are given.

24. The bases of two or more triangles having a common vertex are given, both in magnitude and position, and the sum of the areas is given. Prove that the locus of the vertex is a straight line.

25. If the sum of the perpendiculars from a given point on the sides of a given polygon is given, then the locus of the point is a straight line.

26. If $\triangle ABC$ is an isosceles triangle whose equal sides are AB, AC and if $B'C'$ is any secant cutting the equal sides at B', C', such that $AB' + AC' = AB + AC$, prove that $B'C' > BC$.

27. If A, B are two given points and P is a point on a given line L, prove that the difference between AP and PB is a maximum when L bisects the angle $\angle APB$. Show that their sum is a minimum if it bisects the supplement.

28. Bisect a quadrilateral by a segment constructed from one of its angular points.

29. If AD and BC are two parallel lines cut obliquely by AB and perpendicularly by AC, and between these lines we construct BED, cutting AC at point E such that $ED = 2AB$, prove that the angle $\angle DBC = \frac{1}{3} \cdot \angle ABC$.

30. If O is the point of concurrence of the bisectors of the angles of the triangle $\triangle ABC$, if AO is extended to intersect BC at D, and if OE is constructed from O such that $OE \perp BC$, prove that the $\angle BOD = \angle COE$.

31. The angle made by the bisectors of two consecutive angles of a convex quadrilateral is equal to half the sum of the remaining angles; the angle made by the bisectors of two opposite angles is equal to half the difference of the two other angles.

32. If in the construction of [1.47] we join EF, KG, then $EF^2 + KG^2 = 5AB^2$.

33. Given the midpoints of the sides of a convex polygon of an odd number of sides, construct the polygon.

34. Trisect a quadrilateral by lines constructed from one of its angles.

35. Given the base of a triangle in magnitude and position and the sum of the sides, prove that the perpendicular at either endpoint of the base to the adjacent side and the external bisector of the vertical angle meet on a given line perpendicular to the base.

36. The bisectors of the angles of a convex quadrilateral form a quadrilateral whose opposite angles are supplemental. If the first quadrilateral is a parallelogram, the second is a rectangle; if the first is a rectangle, the second is a square.

37. Suppose that the midpoints of the sides AB, BC, CA of a triangle are respectively D, E, F and that $DG \parallel BF$ and intersects EF. Prove that the sides of the triangle $\triangle DCG$ are respectively equal to the three medians of the triangle $\triangle ABC$.

38. Find the path of a pool ball started from a given point which, after being reflected from the four sides of the table, will pass through another given point. (Assume that the ball does not enter a pocket.)

39. If two lines bisecting two angles of a triangle and terminated by the opposite sides are equal, prove that the triangle is isosceles.

40. State and prove the proposition corresponding to #37 when the base and difference of the sides are given.

41. If a square is inscribed in a triangle, the rectangle under its side and the sum of the base and altitude is equal to twice the area of the triangle.

42. If AB, AC are equal sides of an isosceles triangle and if $BD \perp AC$, prove that $BC^2 = 2AC.CD$.

43. Given the base of a triangle, the difference of the base angles, and the sum or difference of the sides, construct it.

44. Given the base of a triangle, the median that bisects the base, and the area, construct it.

45. If the diagonals AC, BD of a quadrilateral $ABCD$ intersect at E and are bisected at the points F, G, then

$$4 \cdot \triangle EFG = (AEB + ECD) - (AED + EBC)$$

46. If squares are constructed on the sides of any triangle, the lines of connection of the adjacent corners are respectively:

(a) the doubles of the medians of the triangle;

(b) perpendicular to them.

CHAPTER 2

Rectangles

This chapter proves a number of propositions which demonstrate elementary algebraic statements that are more familiar to us in the form of equations. Algebra as we know it had not been developed when Euclid wrote "The Elements". Hence, the results are more of historical importance than practical use except when they appear in subsequent proofs. As such, Book II appears here in truncated form.

Students should feel free to solve the exercises in this chapter algebraically.

Note that Axioms and Mathematical Properties from chapter 1 will not generally be cited.

2.1. Definitions

1. If a point C is taken on a segment AB, point C is the *point of division* between segments AC and CB.

2. If the segment AB is extended to point C, then point C is called a *point of external division*.

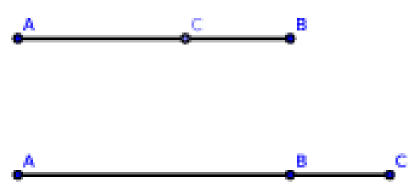

FIGURE 2.1.1. [Def. 2.1] above, [Def 2.2] below

3. A parallelogram whose angles are right angles is called a *rectangle*.

FIGURE 2.1.2. [Def. 2.3 and 2.4]

4. A rectangle is said to be contained by any two adjacent sides. Thus, the rectangle $ABCD$ is said to be contained by AB, AD, or by AB, BC, etc.

5. The rectangle contained by two separate segments (such as AB and CD in Fig 2.1.2) is the parallelogram formed by constructing a perpendicular to AB at A which is equal in length to CD and constructing parallels. The area of the rectangle may also be denoted as $AB.CD$.

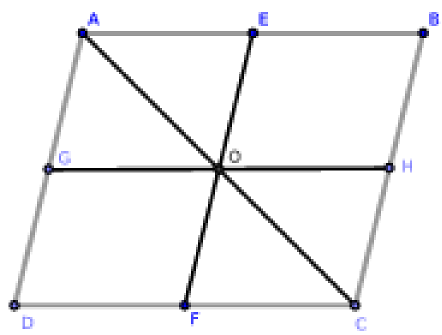

FIGURE 2.1.3. [Def. 2.6]

6. In any parallelogram, a figure which is composed of either of the parallelograms about a diagonal and the two complements is called a *gnomon* [see 1.43]. Thus, if we remove either of the parallelograms $\boxdot AGDE$, $\boxdot OFCH$ from the parallelogram $\boxdot ADCB$, the remainder is a gnomon.

2.2. Axioms

1. A *semicircle* (half-circle) may be constructed given only its center point and a radius.

2.3. Propositions from Book II

PROPOSITION 2.1. *Suppose that two segments (AB, BD) which intersect at one and only one point (B) are constructed such that one segment (BD) is divided into an arbitrary but finite number of segments (BC, CE, EF, FD). Then the rectangle contained by the two segments AB and BD is equal in area to the sum of the areas of the rectangles contained by AB and the subsegments of the divided segment.*

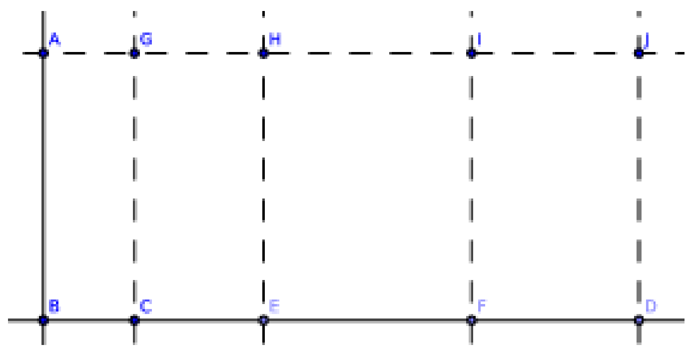

FIGURE 2.3.1. [2.1]

COROLLARY. *1. Algebraically, [2.1] states that the area*

$$AB \cdot BD = AB \cdot BC + AB \cdot CE + AB \cdot EF + AB \cdot FD$$

More generally, it states that if $y = y_1 + y_2 + ... + y_n$, then $xy = xy_1 + xy_2 + ... + xy_n$.

COROLLARY. *2. The rectangle contained by a segment and the difference of two other segments equals the difference of the rectangles contained by the segment and each of the others.*

COROLLARY. *3. The area of a triangle is equal to half the rectangle contained by its base and perpendicular.*

PROPOSITION 2.2. *If a segment (AB) is divided into any two subsegments at a point (C), then the square on the segment is equal to the sum of the rectangles contained by the whole and each of the subsegments (AC, CB).*

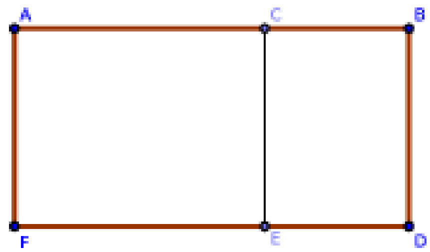

FIGURE 2.3.2. [2.2]

COROLLARY. *1. Algebraically, [2.2] is a special case of [2.1] when* $n = 2$*. Specifically, it states that*

$$AF \cdot FD = AF \cdot FE + AF \cdot ED$$

or if $y = y_1 + y_2$*, then* $xy = xy_1 + xy_2$*.*

PROPOSITION 2.3. *If a segment* (AB) *is divided into two subsegments (at* C*), the rectangle contained by the whole line and either subsegment* $(CB$ *or* $CF)$ *is equal to the square on that segment together with the rectangle contained by each of the segments.*

FIGURE 2.3.3. [2.3]

COROLLARY. *1. Algebraically, [2.3] states that if* $x = y+z$*, then* $xy = y^2+yz$*.*

PROPOSITION 2.4. *If a segment* (AB) *is divided into any two parts (at* C*), the square on the whole segment is equal to the sum of the squares on the subsegments* (AC, CB) *together with twice the area of their rectangle.*

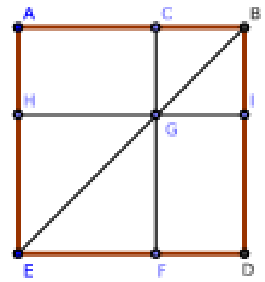

FIGURE 2.3.4. [2.4]

COROLLARY. *1. Algebraically, [2.4] states that if* $x = y + z$*, then* $x^2 = y^2 + 2yz + z^2$.

COROLLARY. *2. The parallelograms about the diagonal of a square are squares.*

COROLLARY. *3. The square on a segment is equal in area to four times the square on its half.*

COROLLARY. *4. If a segment is divided into any number of subsegments, the square on the whole is equal to the sum of the squares on all the subsegments, together with twice the sum of the rectangles contained by the several distinct pairs of subsegments.*

Exercises.

1. Prove [2.4] by using [2.2] and [2.3].

2. If from the vertical angle of a right triangle a perpendicular falls on the hypotenuse, its square equals the area of the rectangle contained by the segments of the hypotenuse.

3. If from the hypotenuse of a right triangle subsegments are cut off equal to the adjacent sides, prove that the square on the middle segment is equal in area to twice the rectangle contained by the segments at either end.

4. In any right triangle, the square on the sum of the hypotenuse and perpendicular from the right angle on the hypotenuse exceeds the square on the sum of the sides by the square on the perpendicular.

5. The square on the perimeter of a right-angled triangle equals twice the rectangle contained by the sum of the hypotenuse and one side and the sum of the hypotenuse and the other side.

PROPOSITION 2.5. *If a segment (AB) is divided into two equal parts (at C) and also into two unequal parts (at D), the rectangle (AD, DB) contained by the unequal parts together with the square on the part between the points of section () is equal to the square on half the line.*

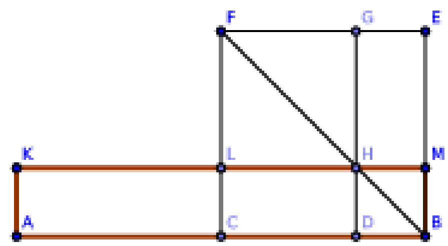

FIGURE 2.3.5. [2.5]

COROLLARY. *1. Algebraically, [2.5] states that*

$$xy = \frac{(x+y)^2}{2} + \frac{(x-y)^2}{2}$$

This may also be expressed as $AD.DB + CD^2 = CB^2 = CA^2$.

COROLLARY. *2. The rectangle $AD.DB$ is the rectangle contained by the sum of the segments AC, CD and their difference, and we have proved it equal to the difference between the square on AC and the square on CD. Hence the difference of the squares on two segments is equal to the rectangle contained by their sum and their difference.*

COROLLARY. *3. The perimeter of the rectangle $AH = 2AB$, and is therefore independent of the position of the point D on the line AB. The area of the same rectangle is less than the square on half the segment by the square on the subsegment between D and the midpoint of the line; therefore, when D is the midpoint, the rectangle will have the maximum area. Hence, of all rectangles having the same perimeter, the square has the greatest area.*

Exercises.

1. Divide a given segment so that the rectangle contained by its parts has a maximum area.

2. Divide a given segment so that the rectangle contained by its subsegments is equal to a given square, not exceeding the square on half the given line.

3. The rectangle contained by the sum and the difference of two sides of a triangle is equal to the rectangle contained by the base and the difference of the segments of the base made by the perpendicular from the vertex.

4. The difference of the sides of a triangle is less than the difference of the segments of the base made by the perpendicular from the vertex.

5. The difference between the square on one of the equal sides of an isosceles triangle and the square on any segment constructed from the vertex to a point in the base is equal to the rectangle contained by the segments of the base.

6. The square on either side of a right triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

PROPOSITION 2.6. *If a segment (AB) is bisected (at C) and extended to a segment (BD), the rectangle contained by the segments (AD, BD) made by the endpoint of the second segment (D) together with the square on half of the segment (CB) equals the square on the segment between the midpoint and the endpoint of the second segment.*

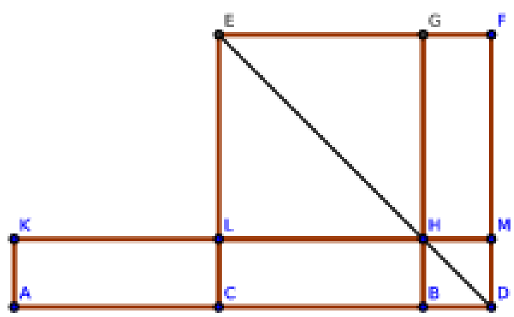

FIGURE 2.3.6. [2.6]

COROLLARY. *1. Algebraically, [2.6] states that*

$$x(x-b) = (x-\frac{b}{2})^2 - (\frac{b}{2})^2$$

This may also be expressed as $AD.BD + CB^2 = CD^2$.

Exercises.

1. Show that [2.6] is reduced to [2.5] by extending the line in the opposite direction.

2. Divide a given segment externally so that the rectangle contained by its subsegments is equal to the square on a given line.

3. Given the difference of two segments and the rectangle contained by them, find the subsegments.

4. The rectangle contained by any two segments equals the square on half the sum minus the square on half the difference.

5. Given the sum or the difference of two lines and the difference of their squares, find the lines.

6. If from the vertex C of an isosceles triangle a segment CD is constructed to any point in the extended base, prove that $CD^2 – CB^2 = AD.DB$.

7. Give a common statement which will include [2.5] and [2.6].

PROPOSITION 2.7. *If a segment (AB) is divided into any two parts (at C), the sum of the squares on the whole segment (AB) and either subsegment (CB) equals twice the rectangle (double AB, CB) contained by the whole segment and that subsegment, together with the square on the remaining segment.*

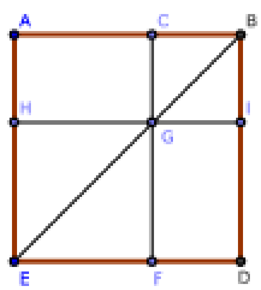

FIGURE 2.3.7. [2.7]

COROLLARY. *1. Algebraically, [2.7] states that if $x = y + z$, then $x^2 + z^2 = 2xz + y^2$; equivalently, this can be stated as $(x - z)^2 = y^2$. Or,*

$$AB^2 + BC^2 = 2 \cdot AB \cdot BC + AC^2$$

COROLLARY. *2. Comparison of [2.4] and [2.7]:*
[2.4]: square on sum = sum of squares + twice rectangle
[2.7]: square on difference = sum of squares-twice rectangle

PROPOSITION 2.8. *If a segment (AB) is divided into two parts (at C), the square on the sum of the whole segment (AB) and either subsegment (BC) equals four times the rectangle contained by the whole line (AB) and that segment, together with the square on the other segment (AC).*

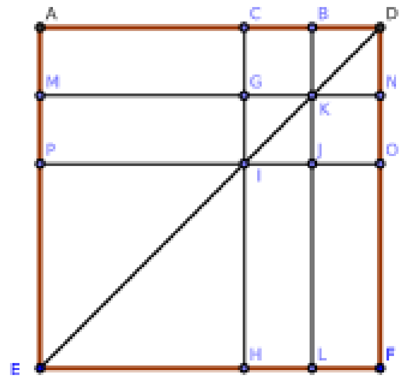

FIGURE 2.3.8. [2.8]

COROLLARY. *1. Algebraically, [2.8] states that if $x = y + z$, then*

$$(x + y)^2 = 4xy + z^2 = 4xy + (x - y)^2$$

Exercises.

1. In [1.47], if EF, GK are joined, prove that $EF^2 – CO^2 = (AB + BO)^2$.

2. Prove that $GK^2 – EF^2 = 3AB \cdot (AO – BO)$.

3. Given that the difference of two segments equals R and the area of their rectangle equals $4R^2$, find the segments.

PROPOSITION 2.9. *If a segment (AB) is bisected (at C) and divided into two unequal segments (at D), the sum of the squares on the unequal subsegments (AD, DB) is double the sum of the squares on half the line (AC) and on the segment (CD) between the points of section.*

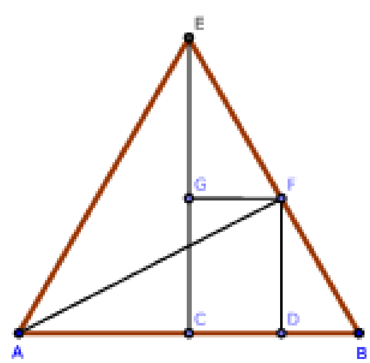

FIGURE 2.3.9. [2.9]

COROLLARY. *Algebraically, [2.9] states that* $(y+z)^2+(y-z)^2=2(y+z)^2$.

Exercises.

1. The sum of the squares on the subsegments of a larger segment of fixed length is a minimum when it is bisected.

2. Divide a given segment internally so that the sum of the squares on the subsegments equals a given square and state the limitation to its possibility.

3. If a segment AB is bisected at C and divided unequally in D, then $AD^2+DB^2=2AD.DB+4CD^2$.

4. Twice the square on the segment joining any point in the hypotenuse of a right isosceles triangle to the vertex is equal to the sum of the squares on the segments of the hypotenuse.

5. If a segment is divided into any number of subsegments, the continued product of all the parts is a maximum and the sum of their squares is a minimum when all the parts are equal.

PROPOSITION 2.10. *If a segment (AB) is bisected (at C) and is extended to a segment (BD), the sum of the squares on the segments (AD, DB) made by the endpoint (D) is equal to twice the square on half the line and twice the square on the segment between the points of that section.*

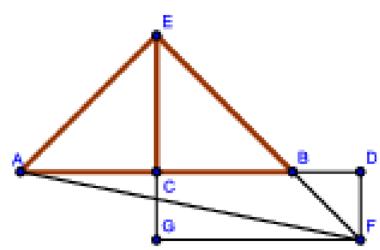

FIGURE 2.3.10. [2.10]

COROLLARY. *1. Algebraically, [2.10] states the same result as Proposition 2.9:* $(y+z)^2+(y-z)^2=2(y+z)^2$.

COROLLARY. *2. The square on the sum of any two segments plus the square on their difference equals twice the sum of their squares.*

COROLLARY. *3. The sum of the squares on any two segments is equal to twice the square on half the sum plus twice the square on half the difference of the lines.*

COROLLARY. *4. If a segment is cut into two unequal subsegments and also into two equal subsegments, the sum of the squares on the two unequal subsegments exceeds the sum of the squares on the two equal subsegments by the sum of the squares of the two differences between the equal and unequal subsegments.*

Exercises.

1. Given the sum or the difference of any two segments and the sum of their squares, find the segments.

2. The sum of the squares on two sides AC, CB of a triangle is equal to twice the square on half the base AB and twice the square on the median which bisects AB.

3. If the base of a triangle is given both in magnitude and position and the sum of the squares on the sides in magnitude, the locus of the vertex is a circle.

4. If in $\triangle ABC$ a point D on the base BC exists such that $BA^2 + BD^2 = CA^2 + CD^2$, prove that the midpoint of AD is equally distant from both B and C.

PROPOSITION 2.11. *It is possible to divide a given segment (AB) into two segments (at H) such that the rectangle (AB, BH) contained by the whole line and one segment is equal in area to the square on the other segment (AH).*

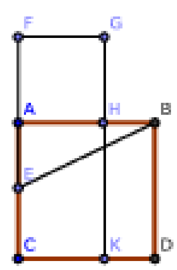

FIGURE 2.3.11. [2.11]

Definition: A segment divided as in this proposition is said to be divided in "extreme and mean ratio."

COROLLARY. *1. Algebraically, [2.11] solves the equation $AB \cdot BH = AH^2$, or $a(a-x) = x^2$. Specifically,*

$$\begin{aligned} a(a-x) &= x^2 \\ a^2 - ax &= x^2 \\ x^2 + ax &= a^2 \\ &\Longrightarrow \\ x &= -\tfrac{a}{2}(1 \pm \sqrt{5}) \end{aligned}$$

Note that $\gamma = \frac{1+\sqrt{5}}{2}$ is called the Golden Ratio[1].

COROLLARY. *2. The segment CF is divided in "extreme and mean ratio" at A.*

COROLLARY. *3. If from the greater segment CA of CF we take a segment equal to AF, it is evident that CA will be divided into parts respectively equal to AH, HB. Hence, if a segment is divided in extreme and mean ratio, the greater segment will be cut in the same manner by taking on it a part equal to the less, and the less will be similarly divided by taking on it a part equal to the difference, and so on.*

FIGURE 2.3.12. [2.11], Cor. 4

COROLLARY. *4. Let AB be divided in "extreme and mean ratio" at C. It is evident ([2.11], Cor. 2) that $AC > CB$. Cut off $CD = CB$. Then by ([2.11], Cor. 2), AC is cut in "extreme and mean ratio" at D, and $CD > AD$. Next, cut off $DE = AD$, and in the same manner we have $DE > EC$, and so on. Since $CD > AD$, it is evident that CD is not a common measure of AC and CB, and therefore not a common measure of AB and AC. Similarly, AD is not a common measure of AC and CD and so is therefore not a common measure of AB and AC. Hence, no matter how far we proceed, we cannot arrive at any remainder which will be a common measure of AB and AC. Hence, the parts of a line divided in "extreme and mean ratio" are incommensurable (i.e., their ratio will never be a rational number).*

[1] https://en.wikipedia.org/wiki/Golden_ratio

See also [6.30] where we divide a given segment (AB) into its "extreme and mean ratio"; that is, we divide a line segment AB at point C such that $AB \cdot BC = AC^2$.

Exercises.

1. The difference between the squares on the segments of a line divided in "extreme and mean ratio" is equal to their rectangle.

2. In a right triangle, if the square on one side is equal to the rectangle contained by the hypotenuse and the other side, the hypotenuse is cut in "extreme and mean ratio" by the perpendicular on it from the right angle.

3. If AB is cut in "extreme and mean ratio" at C, prove that

(a) $AB^2 + BC^2 = 3AC^2$

(b) $(AB + BC)^2 = 5AC^2$

4. The three lines joining the pairs of points G, B; F, D; A, K, in the construction of [2.11] are parallel.

5. If CH intersects BE at O, $AO \perp CH$.

6. If CH is extended, then $CH \perp BF$.

7. Suppose that $\triangle ABC$ is a right-angled triangle having $AB = 2AC$. If AH is equal to the difference between BC and AC, then AB is divided in "extreme and mean ratio" at H.

PROPOSITION 2.12. *On an obtuse-angled triangle ($\triangle ABC$), the square on the side opposite the obtuse angle (AB) exceeds the sum of the squares on the sides (BC, CA) containing the obtuse angle by twice the rectangle contained by either of them (BC) and its extension (CD) to meet a perpendicular (AD) on it from the opposite angle.*

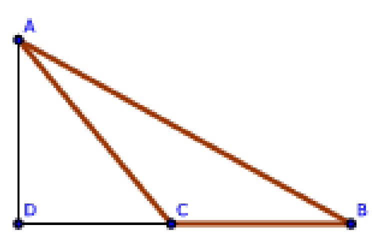

FIGURE 2.3.13. [2.12]

COROLLARY. *1. Algebraically, [2.12] states that in an obtuse triangle $AB^2 = AC^2 + BC^2 + 2 \cdot BC \cdot CD$. This is extremely close to stating the law of cosines: $c^2 = a^2 + b^2 – 2ab \cdot cos(\alpha)$.*

COROLLARY. *2. If perpendiculars from A and B to the opposite sides meet them in H and D, the rectangle $AC.CH$ is equal in area to the rectangle $BC.CD$ (or $\boxdot AC.CH = \boxdot BC.CD$).*

Exercises.

1. If the angle $\angle ACB$ of a triangle is equal to twice the angle of an equilateral triangle, then $AB^2 = BC^2 + CA^2 + BC.CA$.

2. Suppose that $ABCD$ is a quadrilateral whose opposite angles at points B and D are right, and when AD, BC are extended meet at E, prove that $AE.DE = BE.CE$.

3. If $\triangle ABC$ is a right triangle and BD is a perpendicular on the hypotenuse AC, prove that $AB.DC = BD.BC$.

4. If a segment AB is divided at C so that $AC^2 = 2CB^2$, prove that $AB^2 + BC^2 = 2AB.AC$.

5. If AB is the diameter of a semicircle, find a point C in AB such that, joining C to a fixed point D in the circumference and constructing a perpendicular CE intersecting the circumference at E, then $CE^2 – CD^2$ is equal to a given square.

6. If the square of a segment CD, constructed from the angle C of an equilateral triangle $\triangle ABC$ to a point D on the extended side AB is equal to $2AB^2$, prove that AD is cut in "extreme and mean ratio" at B.

PROPOSITION 2.13. *In any triangle ($\triangle ABC$), the square on any side opposite an acute angle (at C) is less than the sum of the squares on the sides containing that angle by twice the rectangle (BC, CD) contained by either of them (BC) and the intercept (CD) between the acute angle and the foot of the perpendicular on it from the opposite angle.*

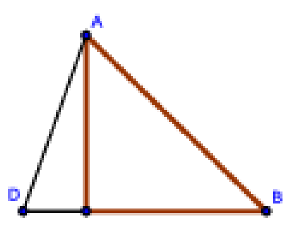

FIGURE 2.3.14. [2.13]

COROLLARY. *1. Algebraically, [2.13] states the same result as [2.12].*

Exercises.

1. If the angle at point C of the $\triangle ACB$ is equal to an angle of an equilateral triangle, then $AB^2 = AC^2 + BC^2 - AC.BC$.

2. The sum of the squares on the diagonals of a quadrilateral, together with four times the square on the line joining their midpoints, is equal to the sum of the squares on its sides.

3. Find a point C in a given extended segment AB such that $AC^2 + BC^2 = 2AC.BC$.

PROPOSITION 2.14. *CONSTRUCTION OF A SQUARE II. It is possible to construct a square equal to a given an arbitrary polygon.*

PROOF. We wish to construct a square equal in area to a given polygon ($MNPQ$).

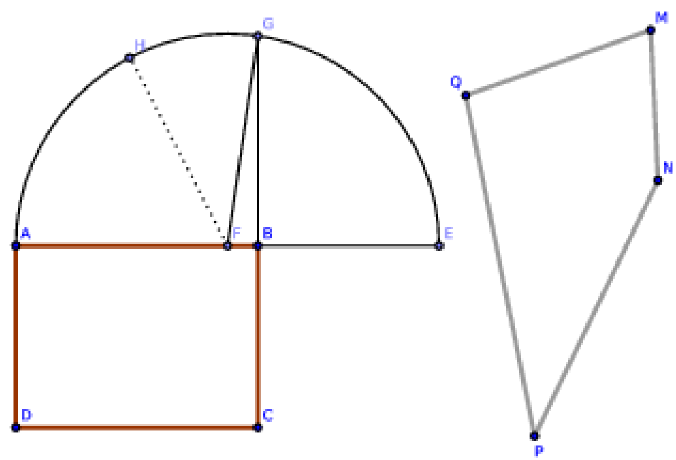

FIGURE 2.3.15. [2.14]

Construct the rectangle $\boxdot ABCD$ equal in area to $MNPQ$ [1.45]. If the adjacent sides of $\boxdot ABCD$ (AB, BC) are equal, $\boxdot ABCD$ is a square and the proof follows.

Otherwise, extend AB to E such that $BE = BC$. Bisect AE at F, and with F as center and FE as radius, construct the semicircle AGE. Extend CB to the semicircle at G. We claim that the square constructed on BG is equal in area to $MNPQ$.

To see this, join FG. Because AE is divided equally at F and unequally at B, $AB.BE + FB^2 = FE^2 = FG^2$ [2.5]. But $FG^2 = FB^2 + BG^2$ by [1.47]. Therefore, the rectangle $AB.BE + FB^2 = FB^2 + BG^2$.

Subtracting FB^2 from both sides of the equality, we have that the rectangle $AB.BE = BG^2$. Since $BE = BC$, the rectangle $AB.BE = \boxdot ABCD$. Therefore $BG^2 = \boxdot ABCD$ which is equal in area to the given polygon, $MNPQ$. □

COROLLARY. *1. The square on the perpendicular from any point on a semicircle to the diameter is equal to the rectangle contained by the segments of the diameter.*

Exercises.

1. Given the difference of the squares on two segments and their rectangle, find the segments.

Examination questions on chapter 2.

1. What is the subject-matter of chapter 2? (Ans. Theory of rectangles.)
2. What is a rectangle? A gnomon?
3. What is a square inch? A square foot? A square mile? (Ans. The square constructed on a line whose length is an inch, a foot, or a mile.)
4. When is a line said to be divided internally? When externally?
5. How is the area of a rectangle determined?
6. How is a line divided so that the rectangle contained by its segments is a maximum?
7. How is the area of a parallelogram found?
8. What is the altitude of a parallelogram whose base is 65 meters and area 1430 square meters?
9. How is a segment divided when the sum of the squares on its subsegments is a minimum?
10. The area of a rectangle is 108.60 square meters and its perimeter is 48.20 linear meters. Find its dimensions.
11. What proposition in chapter 2 expresses the distributive law of multiplication?
12. On what proposition is the rule for extracting the square root founded?
13. Compare [1.47], [2.12], and [2.13].
14. If the sides of a triangle are expressed algebraically by $x^2 + 1$, x^2-1, and $2x$ units, respectively, prove that it is a right triangle.
15. How would you construct a square whose area would be exactly an acre? Give a solution using [1.47].
16. What is meant by incommensurable lines? Give an example from chapter 2.

17. Prove that a side and the diagonal of a square are incommensurable.

18. The diagonals of a lozenge are 16 and 30 meters respectively. Find the length of a side.

19. The diagonal of a rectangle is 4.25 inches, and its area is 7.50 square inches. What are its dimensions?

20. The three sides of a triangle are 8, 11, 15. Prove that it has an obtuse angle.

21. The sides of a triangle are 13, 14, 15. Find the lengths of its medians. Also find the lengths of its perpendiculars and prove that all its angles are acute.

22. If the sides of a triangle are expressed by $m^2 + n^2$, $m^2 - n^2$, and $2mn$ linear units, respectively, prove that it is right-angled.

Chapter 2 exercises.

1. The squares on the diagonals of a quadrilateral are together double the sum of the squares on the segments joining the midpoints of opposite sides.

2. If the medians of a triangle intersect at O, then $AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2)$.

3. Through a given point O, construct three segments OA, OB, OC of given lengths such that their endpoints are collinear and that $AB = BC$.

4. If in any quadrilateral two opposite sides are bisected, the sum of the squares on the other two sides, together with the sum of the squares on the diagonals, is equal to the sum of the squares on the bisected sides together with four times the square on the line joining the points of bisection.

5. If squares are constructed on the sides of any triangle, the sum of the squares on the segments joining the adjacent corners is equal to three times the sum of the squares on the sides of the triangle.

6. Divide a given segment into two parts so that the rectangle contained by the whole and one segment is equal to any multiple of the square on the other segment.

7. If P is any point in the diameter AB of a semicircle and CD is any parallel chord, then $CP^2 + PD^2 = AP^2 + PB^2$.

8. If A, B, C, D are four collinear points taken in order, then $AB.CD + BC.AD = AC.BD$.

9. Three times the sum of the squares on the sides of any pentagon exceeds the sum of the squares on its diagonals by four times the sum of the squares on the segments joining the midpoints of the diagonals.

10. In any triangle, three times the sum of the squares on the sides is equal to four times the sum of the squares on the medians.

11. If perpendiculars are constructed from the angular points of a square to any line, the sum of the squares on the perpendiculars from one pair of opposite angles exceeds twice the rectangle of the perpendiculars from the other pair by the area of the square.

12. If the base AB of a triangle is divided at D such that $mAD = nBD$, then $mAC^2 + nBC^2 = mAD^2 + nDB^2 + (m + n)CD^2$.

13. If the point D is taken on the extended segment AB such that $mAD = nDB$, then $mAC^2 - nBC^2 = mAD^2 - nDB^2 + (m-n)CD^2$.

14. Given the base of a triangle in magnitude and position as well as the sum or the difference of m times the square on one side and n times the square on the other side in magnitude, then the locus of the vertex is a circle.

15. Any rectangle is equal in area to half the rectangle contained by the diagonals of squares constructed on its adjacent sides.

16. If A, B, C, ... are any finite number of fixed points and P a movable point, find the locus of P if $AP^2 + BP^2 + CP^2 + ...$ is given.

17. If the area of a rectangle is given, its perimeter is a minimum when it is a square.

18. Construct equilateral triangles on subsegments AC, CB of segment AB. Prove that if D, D' are the centers of circles constructed about these triangles, then $6DD'^2 = AB^2 + AC^2 + CB^2$.

19. If a, b denote the sides of a right triangle about the right angle and p denotes the perpendicular from the right angle on the hypotenuse, then $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

20. If upon the greater subsegment AB of a segment AC which is divided in extreme and mean ratio, an equilateral triangle $\triangle ABD$ is constructed and CD is joined, then $CD^2 = 2AB^2$.

21. If a variable line, whose endpoints rest on the circumferences of two given concentric circles, stands opposite a right angle at any fixed point, then the locus of its midpoint is a circle.

CHAPTER 3

Circles

Axioms and Mathematical Properties from chapters 1 and 2 will generally not be be cited. This will be a rule that we will apply to subsequent chapters, *mutatis mutandis*.

3.1. Definitions

1. *Equal circles* are those whose radii are equal.

(Note: This is a theorem, and not a definition. If two circles have equal radii, they are evidently congruent figures and therefore equal. Using this method to prove the theorem, [3.26]-[3.29] follow immediately.)

2. A *chord* of a circle is the segment joining two points on its circumference.

3. A segment, ray, or straight line is said to touch a circle when it intersects the circumference of a circle at one and only one point. The segment, ray, or straight line is called a *tangent* to the circle, and the point where it touches the circumference is called the *point of intersection*.

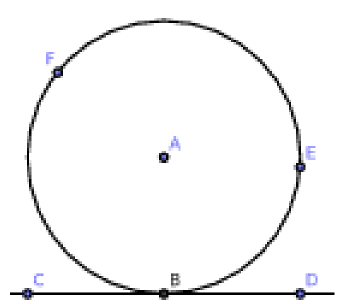

FIGURE 3.1.1. [Def. 3.3] CD touches $\circ FEB$ at B. Or, CD is tangent to $\circ FEB$ and B is the point of intersection between $\circ FEB$ and CD.

(Note: Modern geometry no longer uses Euclid's definitions for curves, tangents, etc. However, it would be far easier to write a new geometry textbook from first principles rather than attempt to update each of Euclid's definitions and begin again[1]. However, Euclid's powerful presentation of complex ideas from simple axioms remains a model for how mathematics should be approached, and students who attempt to master Euclid will find 21st century mathematics more straightforward by doing so.)

4. Circles are said to touch one another when they intersect at one and only one point. There are two types of contact:

a) When one circle is external to the other.

b) When one circle is internal to, or inside, the other.

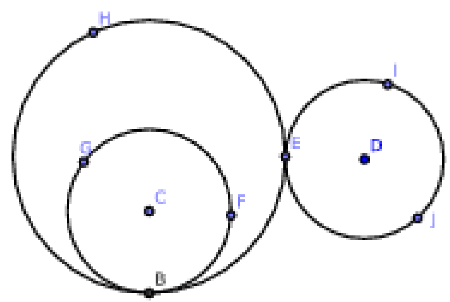

FIGURE 3.1.2. [Def. 3.4] The circles $\circ BEH$ and $\circ IEJ$ touch externally, and the circles $\circ BEH$ and $\circ BFG$ touch internally.

5. A *segment of a circle* is a figure bounded by a chord and one of the arcs into which it divides the circumference.

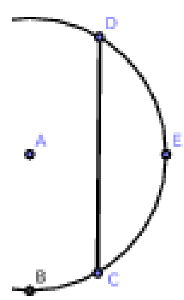

FIGURE 3.1.3. [Def. 3.5] The chord CD of the circle $\circ DEB$ divides the circle into segments DEC and DBC. Segment DEC is bounded by chord CD and arc DEC, and segment DBC is bounded by chord CD and arc DBC.

[1] One such attempt is "The Foundations of Geometry" by David Hilbert, `http://www.gutenberg.org/ebooks/17384`.

6. Chords are said to be equally distant from the center when the perpendiculars constructed to them from the center are equal in length.

7. The angle contained by two lines constructed from any point on the circumference of a segment to the endpoints of its chord is called an *angle in the segment*.

8. The *angle of a segment* is the angle contained between its chord and the tangent at either endpoint.

(Note: A theorem is tacitly assumed in this definition, specifically that the angles which the chord makes with the tangent at its endpoints are equal. We shall prove this later on.)

9. An angle in a segment is said to *stand* on its conjugate arc.

10. Similar segments of circles are those that contain equal angles.

11. A sector of a circle is formed of two radii and the arc included between them.

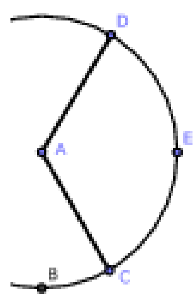

FIGURE 3.1.4. [Def. 3.11] $\circ DEB$ with radii AD and AC forms the sectors $DACE$ and $DACB$.

12. Concentric circles are those which have the same center point.

13. Points which lie on the circumference of a circle are called *concyclic*.

14. A cyclic quadrilateral is one which is inscribed in a circle.

15. A modern definition on an angle[2]:

In geometry, an *angle* is the figure formed by two rays called the sides of the angle which share a common endpoint called the vertex of the angle. This measure is the ratio of the length of a circular arc to its radius, where the arc is centered at the vertex and delimited by the sides.

The size of a geometric angle is usually characterized by the magnitude of the smallest rotation that maps one of the rays into the other. Angles that have the same size are called *congruent angles*.

[2] http://en.wikipedia.org/wiki/Angle

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FIGURE 3.1.5. The measure of angle θ is the quotient of s and r. Author: Gustavb, released under the terms of the GNU Free Documentation License, Version 1.2.

In order to measure an angle θ, a circular arc centered at the vertex of the angle is constructed, e.g. with a pair of compasses. The length of the arc is then divided by the radius of the arc r, and possibly multiplied by a scaling constant k (which depends on the units of measurement that are chosen):

$\theta = ks/r$

The value of θ thus defined is independent of the size of the circle: if the length of the radius is changed, then the arc length changes in the same proportion, and so the ratio s/r is unaltered.

A number of units are used to represent angles: the radian and the degree are by far the most commonly used.

Most units of angular measurement are defined such that one turn (i.e. one full circle) is equal to n units, for some whole number n. In the case of degrees, $n = 360$. A turn of n units is obtained by setting $k = \frac{n}{2\pi}$ in the formula above.

The radian is the angle stands opposite (opposed) by an arc of a circle that has the same length as the circle's radius ($k = 1$ when $k = \frac{n}{2\pi}$). One turn is 2π radians, and one radian is $180/\pi$ degrees, or about 57.2958 degrees. The radian is abbreviated rad, though this symbol is often omitted in mathematical texts, where radians are assumed unless specified otherwise. When radians are used, angles are considered as dimensionless. The radian is used in virtually all mathematical work beyond simple practical geometry, due to the "natural" properties that the trigonometric functions display when their arguments are in radians. The radian is the (derived) unit of angular measurement in the SI system.

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FIGURE 3.1.6. $\theta = s/r$ rad $= 1$ rad. Author: Gustavb, released under the terms of the GNU Free Documentation License, Version 1.2.

The degree, denoted by a small superscript circle (°), is 1/360 of a turn, so one turn is $360°$. Fractions of a degree may be written in normal decimal notation (e.g. $3.5°$ for three and a half degrees), but the "minute" and "second" sexagesimal subunits of the "degree-minute-second" system are also in use, especially for geographical coordinates and in astronomy and ballistics.

Although the definition of the measurement of an angle does not support the concept of a negative angle, it is frequently useful to impose a convention that allows positive and negative angular values to represent orientations and/or rotations in opposite directions relative to some reference.

In a two-dimensional Cartesian coordinate system, an angle is typically defined by its two sides, with its vertex at the origin. The initial side is on the positive x-axis, while the other side or terminal side is defined by the measure from the initial side in radians, degrees, or turns. Positive angles represent rotations toward the positive y-axis, and negative angles represent rotations toward the negative y-axis. When Cartesian coordinates are represented by standard position, defined by the x-axis rightward and the y-axis upward, positive rotations are anticlockwise and negative rotations are clockwise.

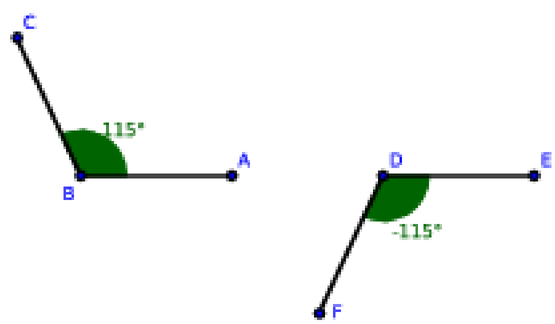

FIGURE 3.1.7. $\angle CBA$ measured as a positive angle, $\angle EDF$ measured as a negative angle

3.2. Propositions from Book III

PROPOSITION 3.1. *THE CENTER OF A CIRCLE I. It is possible to locate the center of a circle.*

PROOF. We wish to find the center of a given circle ($\circ ADB$).

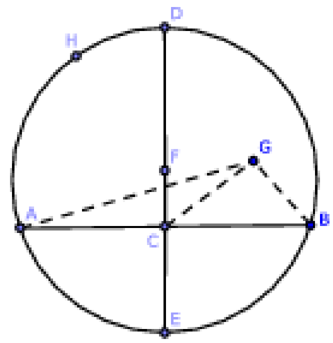

FIGURE 3.2.1. [3.1]

Take any two points A, B in the circumference. Join AB and bisect AB at C. Construct $CD \perp AB$ and extend CD to intersect the circumference at E. Bisect DE at F . We claim that F is the center of $\circ ADB$.

Suppose instead that point G which does not lie on chord DE is the center of $\circ ADB$. Join GA, GC, GB. In the triangles $\triangle ACG$, $\triangle BCG$, we have $AC = CB$ by construction, $GA = GB$ (because they are radii by hypothesis), and the triangles share side CG in common. By [1.8], we have that $\angle ACG = \angle BCG$. Therefore, each angle is a right angle. But $\angle ACD$ is right by construction; therefore $\angle ACG = \angle ACD$ and $\angle ACG = \angle ACD + \angle DCG$, a contradiction.

Hence no point can be the center other than a point on chord DE. By the definition of a circle, it follows that F, the midpoint of DE, must be the center of $\circ ADB$. □

Alternatively:

PROOF. Because ED bisects AB at a right angle, every point equally distant from the points A, B must lie on ED [1.10, #2]. However, the center is also equally distant from A and B. Hence the center must lie on ED. And since it must be equally distant from E and D, it must be the midpoint of ED. □

COROLLARY. *1. The line which bisects any chord of a circle perpendicularly passes through the center of the circle.*

COROLLARY. *2. The locus of the centers of the circles which pass through two fixed points is the line bisecting at right angles the line that connects the two points.*

COROLLARY. *3. If A, B, C are three points on the circumference of a circle, the lines which perpendicularly bisect the chords AB, BC intersect in the center of the circle.*

PROPOSITION 3.2. *POINTS ON A LINE INSIDE AND OUTSIDE A CIRCLE. If any two points are chosen from the circumference of a circle and a line is constructed on these points, then:*

1. The points between the endpoints on the circumference form a chord (i.e., they lie inside the circle).

2. The remaining points of the straight line lie outside the circle.

PROOF. If any two points (A, B) are taken on the circumference of a circle ($\circ FGH$), we claim that:

1. If we construct the line AB, then that segment of the line which lies on and between the points A, B lies within the circle; that is, the line AB contains a segment which is a chord of $\circ FGH$.

2. The remaining points of AB lie outside of the circle.

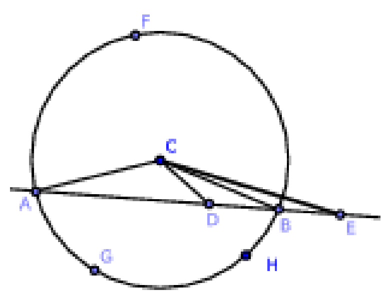

FIGURE 3.2.2. [3.2]

We prove each claim separately:

1. Let C be the center of $\circ FGH$. Take any point D on the segment AB (as opposed to the line AB) and join CA, CD, and CB. Notice that the angle $\angle ADC$ is greater than $\angle ABC$ [1.16]; however, $\angle ABC = \angle CAB$ because $\triangle CAB$ is isosceles [1.5]. Therefore, the angle $\angle ADC > \angle CAB$; it also follows that $\angle ADC > \angle CAD$. Hence, $AC > CD$ [1.29], and so CD is less than the radius of $\circ FGH$. Consequently, the point D must lie within the circle [Def. 1.23]. Similarly, every other point between A and B lies within $\circ FGH$. Finally, since A and B are points in the circumference of $\circ FGH$, AB is a chord.

2. Extend the segment AB in both directions and let E be any point on the extension of AB. Wlog, we assume that E lies closer to B than to A. Join

CE. Then the angle $\angle ABC > \angle AEC$ [1.16]; therefore $\angle CAB > \angle AEC$. Hence $CE > CA$, and so the point E lies outside $\circ FGH$. □

COROLLARY. *1. Three collinear points cannot be concyclic.*

COROLLARY. *2. A straight line, ray, or segment cannot meet a circle at more than two points.*

COROLLARY. *3. The circumference of a circle is everywhere concave towards the center.*

PROPOSITION 3.3. *CHORDS I. Suppose there exist two chords of a circle, one of which passes through the center of the circle. The chord which does not pass through the center is bisected by the chord through the center if and only if the chords are perpendiculars.*

PROOF. Suppose there exist two chords (AB, CD) of a circle ($\circ ACD$), one of which passes through the center of the circle (AB). We claim that the other chord (CD) is bisected if and only if $AB \perp CD$.

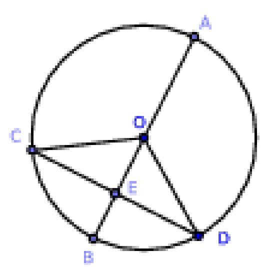

FIGURE 3.2.3. [3.3]

Suppose first that AB bisects CD. We wish to show that $AB \perp CD$.

Let O be the center of $\circ ACD$. Join OC, OD. Then the triangles $\triangle CEO$, $\triangle DEO$ have $CE = ED$ by hypothesis, $OC = OD$ since each are radii of $\circ ACD$, and both triangles have EO in common. By [1.8], $\angle CEO = \angle DEO$; since they are also adjacent angles, each angle is a right angle. Hence $AB \perp CD$.

Now suppose that $AB \perp CD$. We wish to show that AB bisects CD.

Because $OC = OD$, we have that $\angle OCD = \angle ODC$ by [1.5]. Also $\angle CEO = \angle DEO$ by hypothesis, since each angle is right. Therefore, the triangles $\triangle CEO$, $\triangle DEO$ have two angles in one respectively equal to two angles in the other and the side EO common. By [1.26], $CE = ED$. Since $CD = CE + ED$, CD is bisected at E by AB. □

The second part of the proposition may also be proved in this way:

PROOF. By [1.47], we have that

$$\begin{aligned} OC^2 &= OE^2 + EC^2 \\ OD^2 &= OE^2 + ED^2 \end{aligned}$$

Since we also have that $OC^2 = OD^2$, it follows that $EC^2 = ED^2$, and so $EC = ED$. □

COROLLARY. *1. The line which bisects perpendicularly one of two parallel chords of a circle bisects the other perpendicularly.*

COROLLARY. *2. The locus of the midpoints of a system of parallel chords of a circle is the diameter of the circle perpendicular to them all.*

COROLLARY. *3. If a line intersects two concentric circles, its intercepts between the circles are equal in length.*

COROLLARY. *4. The line joining the centers of two intersecting circles bisects their common chord perpendicularly.*

Observation: [3.1], [3.3], and [3.3, Cor. 1] are related such that if any one of them is proved directly, then the other two follow by the Rule of Symmetry.

Exercises.

1. If a chord of a circle stands opposite a right angle at a given point, the locus of its midpoint is a circle.
2. Prove [3.3, Cor. 1].
3. Prove [3.3, Cor. 4].

PROPOSITION 3.4. *CHORDS II. If two chords, one of which is not a diameter, intersect one another in a circle, they do not bisect each another.*

PROOF. In a circle $\circ ACB$, construct two chords (AB, CD) which are not both diameters and intersect each other at a point (E). We claim that AB and CD do not bisect each other.

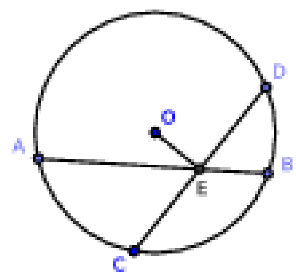

FIGURE 3.2.4. [3.4]

Let O be the center of $\circ ACB$. Since AB, CD are not both diameters, we may join OE.

Suppose that $AE = EB$ and $CE = ED$. Since OE, which intersects the center of the circle, bisects AB, which does not intersect the center of the circle, we must have that $OE \perp AB$. Similarly, $OE \perp CD$. Hence, $\angle AEO = \angle CEO$ where $\angle CEO = \angle CEA + \angle AEO$, a contradiction. Therefore, AB and CD do not bisect each other. $\square$

COROLLARY. *1. If two chords of a circle bisect each other, they are both diameters. (This is the contrapositive statement of [3.4].)*

PROPOSITION 3.5. *NON-CONCENTRIC CIRCLES I. If two circles intersect one another at two points, they are not concentric. (See [Def. 2.13].)*

PROOF. If two circles ($\circ ABC$, $\circ ABD$) intersect at two points (A and B), we claim that the circles are not concentric.

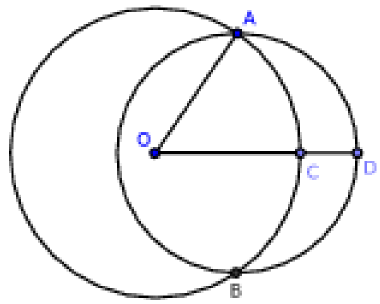

FIGURE 3.2.5. [3.5]

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Suppose that $\circ ABC$, $\circ ABD$ share a common center, O. Join OA and construct a segment OD (where $B \neq D$) which cuts the circles at C and D, respectively. Because O is the center of the circle $\circ ABC$, $OA = OC$. Because O is the center of the circle $\circ ABD$, $OA = OD$. Hence, $OC = OD$ and $OD = OC + CD$, a contradiction. Therefore, $\circ ABC$, $\circ ABD$ are not concentric. □

Exercises.

1. If two non-concentric circles intersect at one point, they must intersect at another point. For let O, O' be the centers of these circles and A be their point of intersection. From A, let AC be the perpendicular on the segment OO'. Extend AC to B, making $BC = CA$. It follows that B is another point of intersection.

2. Two circles cannot have three points in common without coinciding.

PROPOSITION 3.6. *NON-CONCENTRIC CIRCLES II. If one circle intersects another circle internally at one and only one point, then the circles are not concentric.*

PROOF. If a circle ($\circ ABC$) touches another circle ($\circ ADE$) internally at one and only one point (A), then the circles are not concentric.

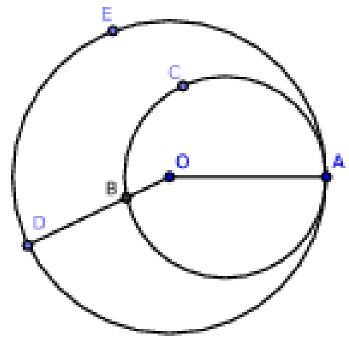

FIGURE 3.2.6. [3.6]

To prove this, suppose that the circles are concentric and let O be the center of each. Join OA, and construct any other segment OD, cutting the circles at the points B, D respectively. Because O is the center of each circle by hypothesis, $OA = OB$ and $OA = OD$; therefore, $OB = OD$ and $OB + BD = OD$, a contradiction. Hence, the circles are not concentric. □

PROPOSITION 3.7. *UNIQUENESS OF SEGMENT LENGTHS FROM A POINT ON THE DIAMETER OTHER THAN THE CENTER. If on the diameter of a circle a point is taken (other than the center of the circle) and from that point segments are constructed to the circumference, the longest segment will contain the center of the circle and the shortest segment will form a diameter with the longest segment. As for the remaining segments, those with endpoints on the circumference nearer to the endpoint on the circumference of the longest segment will be longer than segments with endpoints on the circumference farther from the endpoint of the longest segment. Also, only two equal straight-line segments may be constructed from that point to the circumference, one on each side of the least segment constructed from the given point to circumference.*

PROOF. If from any point (P) on a diameter of a circle (other than the center, O) we construct segments (PA, PB, PC, etc.) to the circumference, one of which passes through the center (PA), we claim that:

1. The longest is the segment which passes through the center (PA).

2. The extension of this segment in the opposite direction (PE) is the shortest segment.

3. Of the others, the segment which is nearest to the segment which passes through the center (PA) is greater than every segment which is more remote (i.e., $PA > PB > PC > PD$).

4. Any two segments making equal angles with the diameter and on opposite sides of the diameter are equal in length (i.e., $PD = PF$).

5. More than two equal segments cannot be constructed from the given point (P) to the circumference.

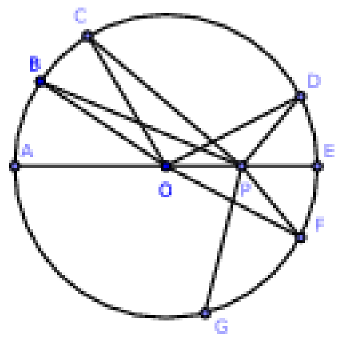

FIGURE 3.2.7. [3.7] $\circ EAG$

We prove each claim separately:

1. The longest is the segment which passes through the center (PA).

Let O be the center of $\circ EAG$. Join OB. Clearly, $OA = OB$. From this we obtain $PA = OA + OP = OB + OP$. Since $OB + OP > PB$ [1.20], it follows that $PA > PB$.

2. The extension of this segment in the opposite direction (PE) is the shortest segment.

Join OD. By [1.20], $OP + PD > OD$. Since, $OD = OE$, it follows that $OP + PD > OE$. Subtracting OP from each side of the inequality, we have that $PD > PE$.

3. Of the others, the segment which is nearest to the segment which passes through the center (PA) is greater than every segment which is more remote (i.e., $PA > PB > PC > PD$).

Join OC. The two triangles $\triangle POB$, $\triangle POC$ have the sides $OB = OC$ with OP in common. But the angle $\angle POB > \angle POC$ since $\angle POB = \angle POC + \angle BOC$. By [1.24], we have that $PB > PC$. Similarly, $PC > PD$.

4. Any two segments making equal angles with the diameter and on opposite sides of the diameter are equal in length (i.e., $PD = PF$).

At O, construct $\angle POF = \angle POD$ and join PF. Consider the triangles $\triangle POD$, $\triangle POF$: each shares side OP, $OD = OF$, and $\angle POD = \angle POF$ by construction. By [1.4], $\triangle POD \cong \triangle POF$, and so $\angle OPF = \angle OPD$ and $PD = PF$. The proof follows.

5. More than two equal segments cannot be constructed from the given point (P) to the circumference.

We claim that a third segment cannot be constructed from P equal to $PD = PF$. Suppose this were possible and let $PG = PD$. Then $PG = PF$; that is, the segment which is nearest to the segment through the center is equal to the one which is more remote; this contradicts point 3, above. Hence, three equal segments cannot is constructed from P to the circumference. □

COROLLARY. *1. If two equal segments PD, PF are constructed from a point P to the circumference of a circle, the diameter through P bisects the angle $\angle DPF$ formed by these segments.*

COROLLARY. *2. If P is the common center of circles whose radii are PA, PB, PC, PD, etc., then:*

(a) The circle whose radius is the maximum segment (PA) lies outside the circle $\circ ADE$ and touches it at A [Def. 3.4].

(b) The circle whose radius is the minimum segment (PE) lies inside the circle $\circ ADE$ and touches it at E.

(c) A circle having any of the remaining radii (such as PD) cuts $\circ ADE$ at two points (D, F).

Observation: [3.7] is a good illustration of the following important definition: if a geometrical magnitude varies its position continuously according to any well-defined relationship, and if it retains the same value throughout, it is said to be a constant (such as the radius of a fixed circle). But if a magnitude increases for some time and then begins to decrease, it is said to be a maximum at the end of the increase. Therefore in the previous figure, PA, which we suppose to revolve around P and meet the circle, is a maximum. Again, if it decreases for some time, and then begins to increase, it is a minimum at the beginning of the increase. Thus PE, which we suppose as before to revolve around P and meet the circle, is a minimum. [3.8] will provide other illustrations of this concept.

PROPOSITION 3.8. *SEGMENT LENGTHS FROM A POINT OUTSIDE THE CIRCLE AND THEIR UNIQUENESS. Suppose a point is chosen outside of a circle and from that point segments are constructed such that they intersect the circumference of the circle at two points, one on the "outer" or convex side of the circumference and one on the "inner" or concave side of the circumference. Let one segment be constructed which intersects the center of the circle and the others all within the same semicircle but not through the center of the circle. Then:*

1. The maximum segment passes through the center.

2. Of the others, those nearer to the segment through the center are greater in length than those which are more remote.

3. If segments are constructed to the convex circumference, the minimum segment is that which passes through the center when extended.

4. Of the other segments, that which is nearer to the minimum is less than one more remote.

5. From the given point outside of the circle, there can be constructed two equal segments to the concave or the convex circumference, both of which make equal angles with the line passing through the center.

6. Three or more equal segments cannot be constructed from the given point outside the circle to either circumference.

PROOF. Construct $\circ ADK$ and point P outside of the circle. We prove each claim separately:

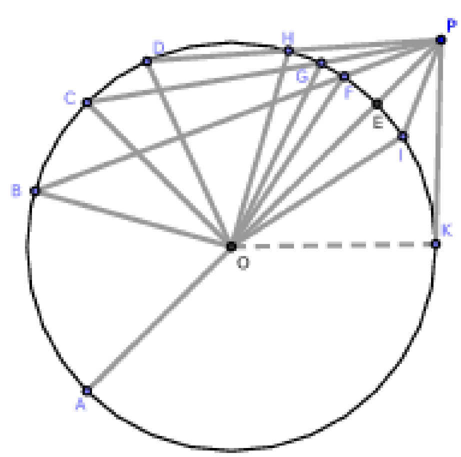

FIGURE 3.2.8. [3.8] $\circ ADK$

1. The maximum segment passes through the center.

Let O be the center of $\circ ADK$ and join OB. Clearly, $OA = OB$. From this we obtain $AP = OA + OP = OB + OP$. But the sum $OB + OP > BP$ [1.20]. Therefore, $AP > BP$.

2. Construct the remaining segments from the figure. Those nearer to the segment through the center are greater in length than those which are more remote.

Join OC, OD. The two triangles $\triangle BOP$, $\triangle COP$ have the side OB equal in length to OC with OP in common, and the angle $\angle BOP > \angle COP$. Therefore, $BP > CP$ [1.24]. Similarly, $CP > DP$, etc.

3. If segments are constructed to the convex circumference, the minimum segment is that which passes through the center when extended.

Join OF. In $\triangle OFP$, the sum $OF + FP > OP = OE + EP$ [1.20]. Recall that $OF = OE$. Subtracting OE and OF from each side of the inequality, we have that $FP > EP$.

4. Of the other segments, that which is nearer to the minimum is less than one more remote.

Join OG, OH. The two triangles $\triangle GOP$, $\triangle FOP$ have two sides GO, OP in one respectively equal to two sides FO, OP in the other, and the angle $\angle GOP > \angle FOP$ By [1.24], $GP > FP$. Similarly, $HP > GP$.

5. From the given point outside of the circle, there can be constructed two equal segments to the concave or the convex circumference, both of which make equal angles with the line passing through the center.

Construct the angles $\angle POI = \angle POF$ [1.23] and join IP. The triangles $\triangle IOP$, $\triangle FOP$ have two sides IO, OP in one respectively equal to two sides FO, OP in the other, and $\angle IOP = \angle FOP$ by construction. By [1.4], $IP = FP$.

6. Three or more equal segments cannot be constructed from the given point outside the circle to either circumference.

A third segment cannot is constructed from P equal to either of the segments IP, FP. If this were possible, let $PK = PF$. Then $PK = PI$, which contradicts part 4, above. □

COROLLARY. *1. If PI is extended to meet the circle at L, then $PL = PB$.*

COROLLARY. *2. If two equal segments are constructed from P to either the convex or concave circumference, the diameter through P bisects the angle between them, and the segments intercepted by the circle are equal in length.*

COROLLARY. *3. If P is the common center of circles whose radii are segments constructed from P to the circumference of $\circ ADK$, then:*

a) The circle whose radius is the minimum segment (PE) has external contact with $\circ ADK$ [Def. 3.4].

b) The circle whose radius is the maximum segment (PA) has internal contact with $\circ ADK$.

c) A circle having any of the remaining segments (PF) as radius intersects $\circ ADK$ at two points (F, I).

PROPOSITION 3.9. *THE CENTER OF A CIRCLE II. A point within a circle from which three or more equal segments can be constructed to the circumference is the center of that circle.*

PROOF. Let D be a point within $\circ ABC$ and from D construct equal segments DA, DB, DC which intersect the circumference. We claim that D is the center of $\circ ABC$.

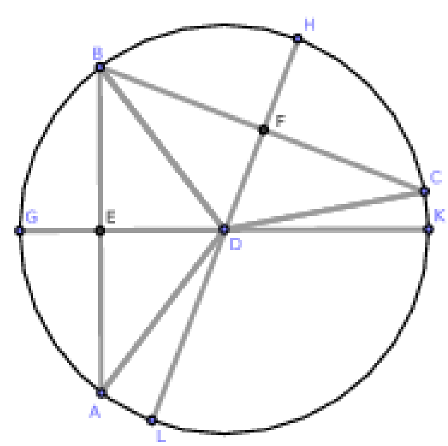

FIGURE 3.2.9. [3.9]

Join AB, BC and bisect them at points E and F, respectively. Join ED and FD and extend these segments to the points G, K, H, and L on the circumference [1.10]. Since $AE = EB$ and ED is a common side, the two sides AE, ED of $\triangle AED$ equal the sides BE, ED of $\triangle BED$; we also have that the base DA

equals the base DB since each are radii of $\circ ABC$. By [1.8], $\triangle AED \cong \triangle BED$, and so $\angle AED = \angle BED$. It follows that $\angle AED$, $\angle BED$ are each right angles.

We have that GK cuts AB perpendicularly into two equal parts. By [3.1, Cor. 1], the center of $\circ ABC$ is a point on GK. Similarly, the center of $\circ ABC$ is a point on HL. Since GK and HL have no other point of intersection except for D, it follows that D is the center of $\circ ABC$. □

Alternatively:

PROOF. Since $AD = LD$, the segment bisecting the angle $\angle ADL$ passes through the center [3.7, Cor. 1]. Similarly, the segment bisecting the angle $\angle BDA$ passes through the center. Hence, the point of intersection of these bisectors, D, is the center. □

PROPOSITION 3.10. *THE UNIQUENESS OF CIRCLES. If two circles have more than two points of their circumferences in common, they coincide.*

PROOF. If two circles ($\circ ABC$, $\circ DAB$) have more than two points of the circumference in common, they coincide.

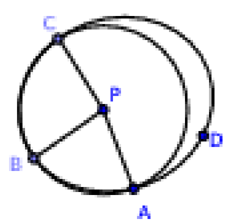

FIGURE 3.2.10. [3.10]

To prove this, suppose that $\circ ABC$, $\circ DAB$ share three points in common (A, B, C) without coinciding. Locate P, the center of $\circ ABC$. Join PA, PB, PC. Since P is the center of $\circ ABC$, we have that $PA = PB = PC$. Again, since $\circ DAB$ is a circle and P a point from which three equal lines PA, PB, PC can be constructed to its circumference, P must be the center of $\circ DAB$ [3.9]. Hence $\circ ABC$ and $\circ DAB$ are concentric, a contradiction. □

COROLLARY. *1. Two circles which do not coincide do not have more than two points common.*

Note: Similarly to [3.10, Cor. 1], two lines which do not coincide cannot have more than one point common.

PROPOSITION 3.11. *SEGMENTS CONTAINING CENTERS OF CIRCLES. If one circle touches another circle internally at one point, then the segment joining the centers of the two circles and terminating on the circumference must have that point of intersection as an endpoint.*

PROOF. If a circle ($\circ CPD$) touches another circle ($\circ APB$) internally at a point (P), we claim that the segment joining the centers of these circles has P as an endpoint.

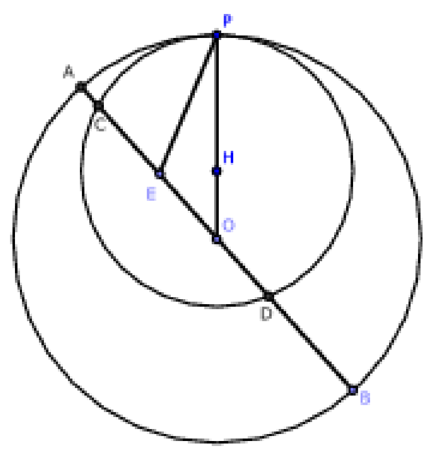

FIGURE 3.2.11. [3.11]

Let O be the center of $\circ APB$ and join OP. We claim that the center of $\circ CPD$ is a point on the segment OP.

Otherwise, let the center of $\circ CPD$ be other than on OP; wlog, choose point E. Join OE, EP, and extend OE through E to meet $\circ CPD$ at C and $\circ APB$ at A. Since E is a point on the diameter of $\circ APB$ between O and A, $EA < EP$ [3.7]. But $EP = EC$ by hypothesis since they are radii of $\circ CPD$. Hence $EA < EC$ and $EA = EC + CA$, a contradiction. Consequently, the center of $\circ APB$ must be on the segment OP.

Hence, the proof. □

Alternatively:

PROOF. Since EP is a segment constructed from a point within the circle $\circ APB$ to the circumference but not forming part of the diameter through E, the circle whose center is E with radius EP cuts $\circ APB$ at P [3.7, Cor. 2] and also touches it at P by hypothesis, a contradiction. A similar argument holds for all points not on the segment OP. Hence, the center of $\circ CPD$ must be on OP. □

PROPOSITION 3.12. *INTERSECTING CIRCLES I. If two circles intersect externally at one point, then the segment joining their centers passes through that point.*

PROOF. If two circles ($\circ PCF$, $\circ PDE$) have external contact at the point P, we claim that the segment joining their centers intersects P.

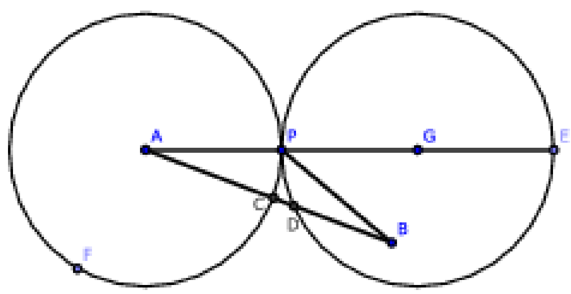

FIGURE 3.2.12. [3.12]

Let A be the center of $\circ PCF$. Join AP and extend it to meet $\circ PDE$ at E. We claim that the center of $\circ PDE$ is on the segment PE.

Otherwise, let it be at B. Join AB, intersecting $\circ PCF$ at C and $\circ PDE$ at D, and join BP. Since A is the center of the $\circ PCF$, $AP = AC$; and since B is the center of $\circ PDE$, $BP = BD$. Hence the sum $AP + BP = AC + DB$. But $AB > AC + DB$. Therefore, $AB > AP + PB$, and one side of $\triangle APB$ is greater than the sum of the other two, a contradiction [1.20].

Hence the center of $\circ PDE$ must be on the segment PE. Let it be G, and the proof follows. □

Alternatively:

PROOF. Suppose that the center of $\circ PDE$ lies on the segment BP. Since BP is a segment constructed from a point outside of the circle $\circ PCF$ to its circumference which does not pass through the center when it is extended, the circle whose center is B with radius BP must cut the circle $\circ PCF$ at P [1.8, Cor. 3]. However, it touches $\circ PCF$ at P by hypothesis, a contradiction. Since BP was chosen arbitrarily, the center of $\circ PDE$ must lie on the segment PE. □

Observation: [3.11] and [3.12] may both be included in one theorem: "If two circles touch each other at any point, the centers and that point are collinear." This is a limiting case of the theorem given in [3.3, Cor. 4]: "The line joining the centers of two intersecting circles bisects the common chord perpendicularly."

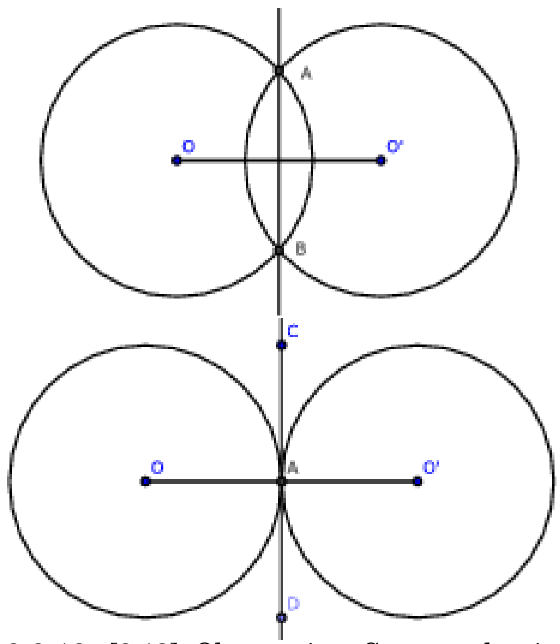

FIGURE 3.2.13. [3.12], Observation. Suppose the circles with centers O and O' have two points of intersection, A and B. Suppose further that A remains fixed while the second circle moves so that the point B ultimately coincides with A. Since the segment OO' always bisects AB, we see that OO' intersects A. In consequence of this motion, the common chord CD becomes the tangent to each circle at A.

COROLLARY. *1. If two circles touch each other, their point of intersection is the union of two points of intersection. When counting the number of points at which two circles intersect, we may for purposes of calculation consider this point of intersection as two points. See Cor. 2 for details.*

COROLLARY. *2. If two circles touch each other at any point, they cannot have any other common point.*

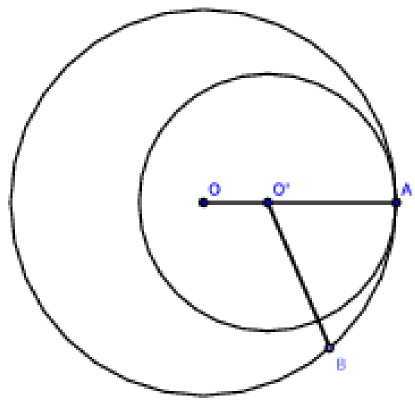

FIGURE 3.2.14. [3.2, Cor. 2]

For, since two circles cannot have more than two points common [3.10] and their point of intersection is equivalent to two points for purposes of calculation, circles that touch cannot have any other point common. The following is a formal proof of this Corollary:

Let O, O' be the centers of the two circles where A is the point of intersection, and let O' lie between O and A. Take any other point B in the circumference of O, and join $O'B$. By [3.7], $O'B > O'A$. Therefore, B is outside the circumference of the inner circle. Hence, B cannot be common to both circles. Since point B was chosen arbitrarily, the circles cannot have any other common point except for A.

PROPOSITION 3.13. *INTERSECTING CIRCLES II. Two circles cannot touch each other at two points either internally or externally.*

PROOF. We divide the proof into its internal and external cases:

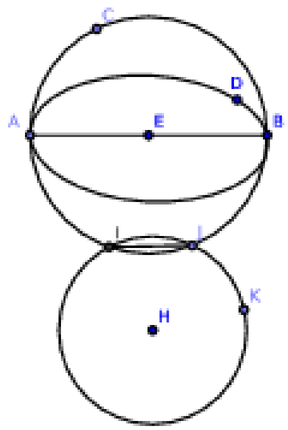

FIGURE 3.2.15. [3.13]

1. Suppose two distinct circles $\circ ACB$ and $\circ ADB$ touch internally at two points A and B. Since the two circles touch at A, the segment joining their centers passes through A [3.11]. Similarly, the segment joining their centers passes through B. Hence, the centers of these circles and the points A, B are on one segment, and so AB is a diameter of each circle. Hence, if AB is bisected at E, E must be the center of each circle, i.e., the circles are concentric, a contradiction [1.5].

2. If two circles $\circ ACB$ and $\circ IJK$ touch externally at points I and J, then by [3.12] the segment joining the centers of $\circ ACB$ and $\circ IJK$ passes through the centers E and H as well as points I and J, a contradiction. □

An alternative proof to part 1:

PROOF. Construct a line bisecting AB perpendicularly. By [3.1, Cor. 1], this line passes through the center of each circle, and by [3.11], [3.12] must pass through each point of intersection, a contradiction. Hence, two circles cannot touch each other at two points. □

This proposition is an immediate inference from [3.12, Cor. 1] that a point of intersection counts for two intersections, for then two contacts would be equivalent to four intersections; but there cannot be more than two intersections [3.10]. It also follows from [3.12, Cor. 2] that if two circles touch each other at point A, they cannot have any other point common. Hence, they cannot touch again at B.

Exercises.

1. If a circle with a non-fixed center touches two fixed circles externally, the difference between the distances of its center from the centers of the fixed circles is equal to the difference or the sum of their radii, according to whether the contacts are of the same or of opposite type [Def. 3.4].

2. If a circle with a non-fixed center is touched by one of two fixed circles internally and touches the other fixed circle either externally or internally, the sum of the distances from its center to the centers of the fixed circles is equal to the sum or the difference of their radii, according to whether the contact with the second circle is internal or external.

3. Suppose two circles touch externally. If through the point of intersection any secant is constructed cutting the circles again at two points, the radii constructed to these points []e parallel.

4. Suppose two circles touch externally. If two diameters in these circles are parallel, the line from the point of intersection to the endpoint of one diameter passes through the endpoint of the other.

5. Rewrite the results of #3 and #4.

PROPOSITION 3.14. *EQUALITY OF CHORD LENGTHS. Chords in a circle are equal in length if and only if they are equally distant from the center.*

PROOF. In circles with equal radii, we claim that:

1. chords of equal length (AB, CD) are equally distant from the center (O).

2. chords which are equally distant from the center are also equal in length.

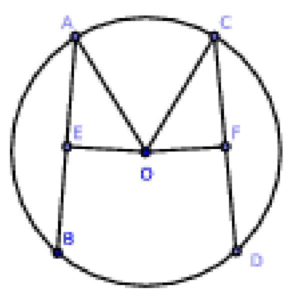

FIGURE 3.2.16. [3.14]

Let O be the center of $\circ ACD$, and construct the chords AB, CD. We prove each claim separately:

1. Let $AB = CD$ and construct perpendicular segments OE, OF. We wish to prove that $OE = OF$.

Join AO, CO. Because AB is a chord in a circle and OE is a segment constructed from the center where $OE \perp AB$, OE bisects AB [3.3]. It follows that $AE = EB$. Similarly, $CF = FD$. But $AB = CD$ by hypothesis, and so $AE = CF$.

Because $\angle OEF$ is a right angle, $AO^2 = AE^2 + EO^2$. Similarly, $CO^2 = CF^2 + FO^2$. Since $AO^2 = CO^2$, we have that $AE^2 + EO^2 = CF^2 + FO^2$ where $AE^2 = CF^2$. Hence $EO^2 = FO^2$, and so $EO = FO$. Hence AB, CD are equally distant from O [Def. 3.6].

2. Now let $EO = FO$. We wish to prove that $AB = CD$.

As before, we have $AE^2 + EO^2 = CF^2 + FO^2$ where $EO^2 = FO^2$ by hypothesis. Hence $AE^2 = CF^2$, and so $AE = CF$. But $AB = 2 \cdot AE$ and $CD = 2 \cdot CF$, from which it follows that $AB = CD$. □

Exercise.

1. If a chord of given length slides around a fixed circle, then:

a) the locus of its midpoint is a circle;

b) the locus of any point fixed on the chord is a circle.

PROPOSITION 3.15. *INEQUALITY OF CHORD LENGTHS. The diameter is the longest chord in a circle, and a chord is nearer to the center of a circle than another chord if and only if it is the longer of the two chords.*

PROOF. Construct $\circ ACB$ with center O, diameter AB, and chords CD, EF. We claim that:

1. The diameter (AB) is the longest chord in a circle;

2. A chord which is nearer to the center (CD) is longer than a chord which is more distant (EF);

3. Longer chords are nearer to the center than shorter chords.

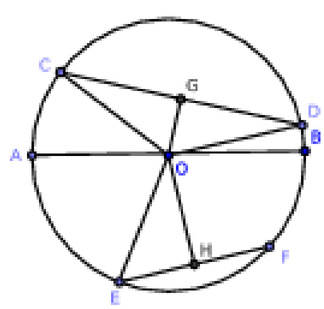

FIGURE 3.2.17. [3.15]

We prove each claim separately:

1. The diameter (AB) is the longest chord in a circle.

Join OC, OD, OE and construct the perpendiculars OG, OH. Because O is the center of $\circ AEF$, $OA = OC$ and $OB = OD$. Hence $AB = OA + OB = OC + OD$. But $OC + CD > CD$ by [1.20]. Therefore, $AB > CD$.

2. A chord which is nearer to the center (CD) is longer than a chord which is more distant (EF).

Suppose that CD is nearer to O than EF. It follows that $OG < OH$ [3.14]. Since $\triangle OGC$ and $\triangle OHE$ are right triangles, we have that $OC^2 = OG^2 + GC^2$ and $OE^2 = OH^2 + HE^2$. Since $OC = OE$, $OG^2 + GC^2 = OH^2 + HE^2$. But $OG^2 < OH^2$, and so $GC^2 > HE^2$. It follows that $GC > HE$. Since $CD = 2 \cdot GC$ and $EF = 2 \cdot HE$, it follows that $CD > EF$.

3. Longer chords are nearer to the center than shorter chords.

Suppose that $CD > EF$. We wish to prove that $OG < OH$.

As before, we have $OG^2 + GC^2 = OH^2 + HE^2$. By our hypothesis, we have that $GC^2 > HE^2$. Therefore $OG^2 < OH^2$, and so $OG < OH$. □

Exercises.

1. The shortest chord which can be constructed through a given point within a circle is the perpendicular to the diameter which passes through that point.

2. Through a given point, within or outside of a given circle, construct a chord of length equal to that of a given chord.

3. Through one of the points of intersection of two circles, construct a secant

a) where the sum of its segments intercepted by the circles is a maximum;

b) which is of any length less than that of the maximum.

4. Suppose that circles touch each other externally at A, B, C and that the chords AB, AC of two of them are extended to meet the third again in the points D and E. Prove that DE is a diameter of the third circle and is parallel to the segment joining the centers of the others.

PROPOSITION 3.16. *THE PERPENDICULAR TO A DIAMETER OF A CIRCLE. The perpendicular to a diameter of a circle intersects the circumference at one and only one point, and any other segment through the diameter's endpoint intersects the circle at two points.*

PROOF. Construct $\circ DAB$ with center C. We claim that:

1. The perpendicular (BI) to the diameter (AB) of $\circ DAB$ touches the circle only at their point of intersection.

2. Any other line or segment (BH) through the same point (B) cuts the circle.

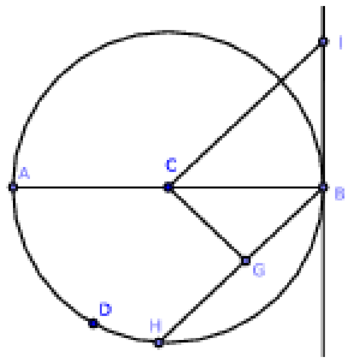

FIGURE 3.2.18. [3.16]

We prove each claim separately:

1. The perpendicular (BI) to the diameter (AB) of $\circ DAB$ touches the circle only at their point of intersection.

Let I be an arbitrary point on BI and construct the segment CI. Because $\angle CBI$ is a right angle, $CI^2 = CB^2 + BI^2$ [1.47]. It follows that $CI^2 > CB^2$, and so $CI > CB$. This confirms that I lies outside of the circle $\circ DAB$ [3.2]. Similarly, every other point on BI except B lies outside of the $\circ DAB$. Hence, since BI meets the circle $\circ DAB$ at B and does not cut it, BI must touch $\circ DAB$ at B.

2. Any other line or segment (BH) through the same point (B) cuts the circle.

We wish to prove that BH, which is not perpendicular to AB, cuts the circle. Construct $CG \perp HB$. It follows that $BC^2 = CG^2 + GB^2$. Therefore $BC^2 > CG^2$, and so $BC > CG$. Hence the point G must be within the circle [3.2], and consequently if the segment BG is extended it must meet $\circ DAB$ at H and therefore cut it. □

Exercises.

1. If two circles are concentric, all chords of the greater circle which touch the lesser circle are equal in length.

2. Construct a parallel to a given line which touches a given circle.

3. Construct a perpendicular to a given line which touches a given circle.

4. Construct a circle having its center at a given point

   a) and touches a given line;

   b) and touches a given circle.

How many solutions exist in this case?

5. Construct a circle of given radius that touches two given lines. How many solutions exist?

6. Find the locus of the centers of a system of circles touching two given lines.

7. Construct a circle of given radius that touches a given circle and a given line or that touches two given circles.

PROPOSITION 3.17. *TANGENTS ON CIRCLES I. It is possible to construct a tangent of a given circle from a given point outside of the circle.*

PROOF. We wish to construct a tangent to a given circle ($\circ BCD$) from a given point (P) outside of the circle.

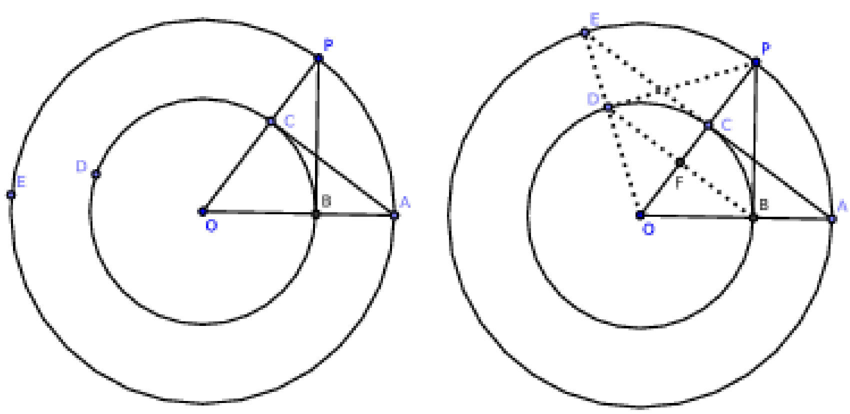

FIGURE 3.2.19. [3.17] (α), (β)

Let O be the center of $\circ BCD$ (Fig. 3.2.19). Join OP, cutting the circumference at C. With O as center and OP as radius, construct the circle $\circ EAP$. Also construct $CA \perp OP$. Join OA, intersecting $\circ BCD$ at B, and join BP. We claim that BP is the required tangent to $\circ BCD$.O

Since O is the center of $\circ CDB$ and $\circ EAP$, we have that $OA = OP$ and $OC = OB$. Hence $\triangle AOC$, $\triangle POB$ have the sides OA, OC in one triangle respectively equal to the sides OP, OB in the other with the contained angle common to both. By [1.4], $\triangle AOC \cong \triangle POB$, and so $\angle OCA = \angle OBP$. But $\angle OCA$ is a right angle by construction. Therefore $\angle OBP$ is a right angle, and so by [3.16], PB touches the circle $\circ BCD$ at B. Hence, PB is a tangent of $\circ BCD$ at point B. $\square$

COROLLARY. *1. If AC in Fig. 3.2.26(β) is extended to E, OE is joined, the circle $\circ BCD$ is cut at D, and the segment DP is constructed, then DP is another tangent of $\circ BCD$ (at point P).*

Exercises.

1. The two tangents PB, PD (in Fig. 3.2.26(β)) are equal in length to one another because the square of each is equal to the square of OP minus the square of the radius.

2. If a quadrilateral is circumscribed to a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

3. If a parallelogram is circumscribed to a circle, it must be a lozenge, and its diagonals intersect at the center.

4. In Fig. 3.2.26(β), if BD is joined and OP is intersected at F, then $OP \perp BD$.

5. The locus of the intersection of two equal tangents to two circles is a segment (called the *radical axis* of the two circles).

6. Find a point such that tangents from it to three given circles is equal. (This point is called the *radical center* of the three circles.)

7. The rectangle $OF.OP$ is equal in area to the square of the radius of $\circ BCD$.

Definition: Suppose we have two points F and P such that when the area of the rectangle $OF.OP$ (where O is the center of a given circle) is equal to the area of the square of the radius of that circle, then F and P are called *inverse points* with respect to the circle.

9. The intercept made on a variable tangent by two fixed tangents stands opposite a constant angle at the center.

10. Construct a common tangent to two circles. Demonstrate how to construct a segment cutting two circles so that the intercepted chords are of given lengths.

PROPOSITION 3.18. *TANGENTS ON CIRCLES II. If a line touches a circle, the segment from the center of the circle to the point of intersection with the line is perpendicular to the line.*

PROOF. If the line CD touches $\circ ABC$, we claim that the segment OC from the center (O) to the point of intersection (C) is perpendicular to CD.

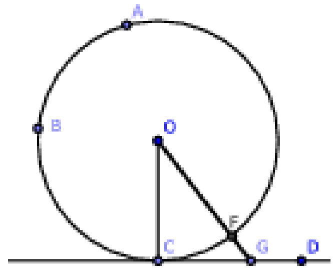

FIGURE 3.2.20. [3.18]

Otherwise, suppose that another segment OG is constructed from the center such that $OG \perp CD$. Let OG cut the circle at F. Because the angle $\angle OGC$ is right by hypothesis, the angle $\angle OCG$ must be acute [1.17]. By [1.19], $OC > OG$. But $OC = OF$, and therefore we have that $OF > OG$ and $OG = OF + FG$, a contradiction. Hence $OC \perp CD$. □

Alternatively:

PROOF. Since the perpendicular must be the shortest segment from O to CD and OC is evidently the shortest line, it follows that $OC \perp CD$. □

PROPOSITION 3.19. *TANGENTS ON CIRCLES III. If a line is a tangent to a circle, then the perpendicular constructed from its point of intersection passes through the center of the circle.*

PROOF. If a line (AB) is tangent to a circle ($\circ CDA$), we claim that the perpendicular (AC) constructed from its point of intersection (A) passes through the center of $\circ CDA$.

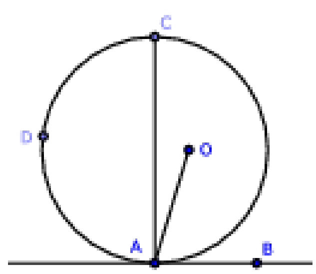

FIGURE 3.2.21. [3.19]

Suppose it were otherwise; let O be the center of $\circ CDA$ and join AO. Because the line AB touches $\circ CDA$ and OA is constructed from the center to the point of intersection, $OA \perp AB$ [3.18]. Therefore $\angle OAB$ and $\angle CAB$ are right angles. It follows that $\angle OAB = \angle CAB$ and $\angle CAB = \angle OAB + \angle OAC$, a contradiction. Hence, the center must lie on the segment AC. □

COROLLARY. *1. If a number of circles touch the same line at the same point, the locus of their centers is the perpendicular to the line at the point.*

COROLLARY. *2. Suppose we have a circle and any two of the following properties:*

a) a tangent to a circumference;

b) a segment, ray, or straight line constructed from the center of the circle to the point of intersection;

c) right angles at the point of intersection.

Then by [3.16], [3.18], [3.19], and the Rule of Symmetry, the remaining property follows. If we have (a) and (c), then it may be necessary to extend a given segment or a ray to the center of the circle. These are limiting cases of [3.1, Cor. 1] and [3.3].

PROPOSITION 3.20. *ANGLES AT THE CENTER OF A CIRCLE AND ON THE CIRCUMFERENCE. The angle at the center of a circle is double the angle at the circumference when each stands on the same arc of the circumference.*

PROOF. Construct $\circ ABC$ with center O and radius OB as in Fig. 3.2.22 (α).

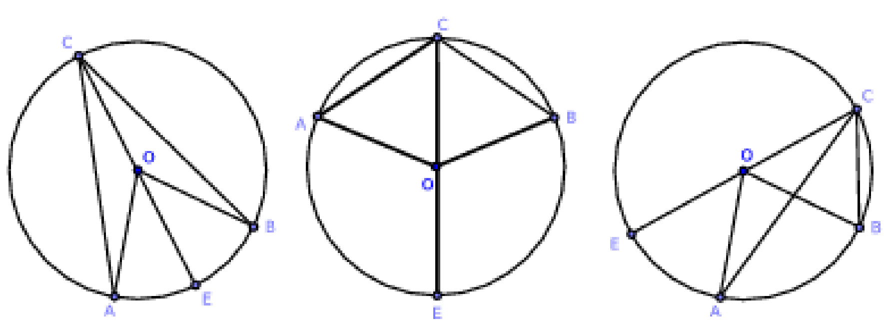

FIGURE 3.2.22. [3.20], (α), (β), (γ)

Construct $\angle AOB$ at its center and $\angle ACB$ at the circumference such that each angle stands on the arc AB. We claim that $\angle AOB = 2 \cdot \angle ACB$.

Join CO and extend CO through to the circumference at E. Since $OC = OA$, we have that $\angle OCA = \angle OAC$. By [1.5], $\angle OCA + \angle OAC = 2 \cdot \angle OCA$. Also, $\angle AOE = \angle OCA + \angle OAC$ [1.32]. It follows that $\angle AOE = 2 \cdot \angle OCA$. Similarly, $\angle EOB = 2 \cdot \angle OCB$, and so we have that

$$\begin{aligned}\angle AOB &= \angle AOE + \angle EOB \\ &= 2 \cdot \angle OCA + 2 \cdot \angle OCB \\ &= 2 \cdot \angle ACB\end{aligned}$$

Now construct $\circ ABC$ with center O and radius OB with $\angle ACB$ as in Fig. 3.2.22 (γ). Join CO through to the circumference at E. Similarly to the above, we can show that $\angle EOB = 2 \cdot \angle OCB$ and $\angle EOA = 2 \cdot \angle ECA$. It follows that

$$\begin{aligned}\angle EOB - \angle EOA &= \angle AOB \\ &= 2 \cdot \angle OCB - 2 \cdot \angle ECA \\ &= 2 \cdot (\angle OCB - \angle ECA) \\ &= 2 \cdot \angle ACB\end{aligned}$$

Therefore, the angle at the center of a circle is double the angle at the circumference when the angles stand on the same arc. □

Alternatively:

PROOF. Construct $\circ ABC$ as in Fig. 3.2.2(α), $\circ ACB$ as in Fig. 3.2.2(β), and $\circ ECB$ in Fig. 3.22(γ).

Join CO and extend it to E. Because $OA = OC$, it follows that $\angle ACO = \angle OAC$. Since $\angle AOE = \angle OAC + \angle ACO$ [1.32], we have that $\angle AOE = 2 \cdot \angle ACO$. Similarly, $\angle EOB = 2 \cdot \angle OCB$. Hence (by adding in the cases of Fig 3.2.22 (α), (β), and subtracting in (γ)), we have that $\angle AOB = 2 \cdot \angle ACB$. □

COROLLARY. *1. If AOB is a line, then $\angle ACB$ is a right angle; specifically, the angle in a semicircle is a right angle (compare with [3.31]).*

PROPOSITION 3.21. *ANGLES ON CHORDS. In a circle, angles standing on the same arc are equal in measure to one another.*

PROOF. We claim that the angles ($\angle ACB$, $\angle ADB$) standing within $\circ ABC$ and on the same arc (AB) are equal in measure.

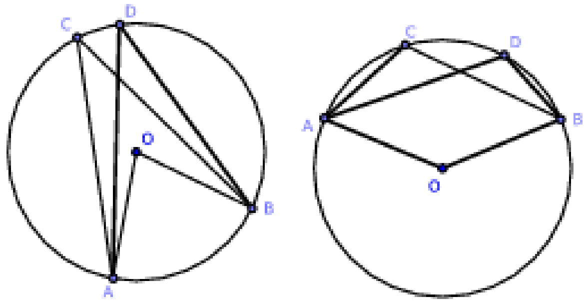

FIGURE 3.2.23. [3.21], (α), (β)

We first consider the cases presented in Fig. 3.2.23(α), (β). Let O be the center of $\circ ABC$, and join OA, OB. By [3.20], $\angle AOB = 2 \cdot \angle ACB$ and $\angle AOB = 2 \cdot \angle ADB$. It follows that $\angle ACB = \angle ADB$.

We now consider the case presented in Fig. 3.2.24(γ):

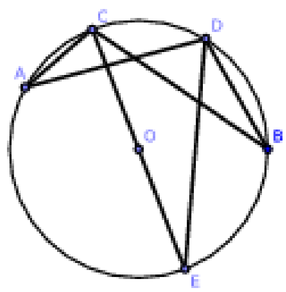

FIGURE 3.2.24. [3.21] (γ)

Let O remain the center of $\circ ABC$. Join CO and extend the segment to intersect the circumference of $\circ CAB$ at E. Join DE. Since O is the center, the arc ACE is greater than a semicircle; similarly to our first case, we obtain that $\angle ACE = \angle ADE$ and $\angle ECB = \angle EDB$. Hence, we have that

$$\begin{aligned} \angle ACB &= \angle ACE + \angle ECB \\ &= \angle ADE + \angle EDB \\ &= \angle ADB \end{aligned}$$

□

COROLLARY. *1. If two triangles $\triangle ACB$, $\triangle ADB$ stand on the same base AB and have equal vertical angles on the same side of it, the four points A, C, D, B are concyclic.*

COROLLARY. *2. If A, B are two fixed points and if C varies its position in such a way that the angle $\angle ACB$ retains the same value throughout, the locus of C is a circle. (Or: given the base of a triangle and the vertical angle, the locus of the vertex is a circle).*

Exercises.

1. Given the base of a triangle and the vertical angle, find the locus

(a) of the intersection of its perpendiculars;

(b) of the intersection of the internal bisectors of its base angles;

(c) of the intersection of the external bisectors of the base angles;

(d) of the intersection of the external bisector of one base angle and the internal bisector of the other.

2. If the sum of the squares of two segments is given, their sum is a maximum when the segments are equal in length.

3. Of all triangles having the same base and vertical angle, the sum of the sides of an isosceles triangle is a maximum.

4. Of all triangles inscribed in a circle, the equilateral triangle has the maximum perimeter.

5. Of all concyclic figures having a given number of sides, the area is a maximum when the sides are equal.

PROPOSITION 3.22. *QUADRILATERALS INSCRIBED INSIDE CIRCLES. The sum of the opposite angles of a quadrilateral inscribed in a circle equals two right angles.*

PROOF. We claim that the sum of the opposite angles of a quadrilateral ($ABCD$) inscribed in a circle ($\circ CBA$) equals two right angles.

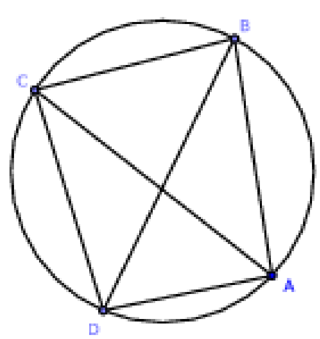

FIGURE 3.2.25. [3.22]

Join AC, BD. Since $\angle ABD$ and $\angle ACD$ stand on the same arc AD, we have that $\angle ABD = \angle ACD$ [3.21]. Similarly, $\angle DBC = \angle DAC$ because they stand on the arc DC. Hence, $\angle ABC = \angle ACD + \angle DAC$. From this, we obtain

$$\angle ABC + \angle CDA = \angle ACD + \angle DAC + \angle CDA$$

where the right-hand side of the equality is the sum of the three angles of $\triangle ACD$. Since this sum equals two right angles [1.32], we have that $\angle ABC + \angle CDA$ equals two right angles. We obtain an analogous result for $\angle DAB + \angle BCD$. □

Alternatively:

PROOF. Let O be the center of $\circ CBA$.

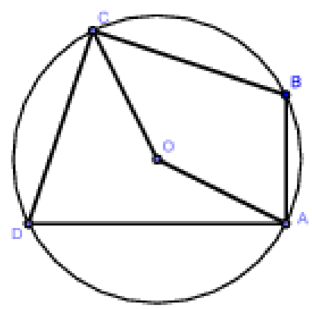

FIGURE 3.2.26. [5.22], alternative proofs

Join OA, OC. Define $\angle AOC$ as less than two right angles and $\angle COA$ as more than two right angles. Also, notice that $\angle AOC + \angle COA$ =four right angles.

Notice that $\angle AOC = 2 \cdot \angle CDA$ and $\angle COA = 2 \cdot \angle ABC$ by [3.20]. Hence $\angle AOC + \angle COA = 2 \cdot (\angle CDA + \angle ABC)$. Since, $\angle AOC + \angle COA$ equals four right angles, $\angle CDA + \angle ABC$ equals two right angles. □

COROLLARY. *1. If the sum of two opposite angles of a quadrilateral are equal to two right angles, then a circle may be inscribed about the quadrilateral.*

COROLLARY. *2. If a parallelogram is inscribed in a circle, then it is a rectangle.*

Exercises.

1. If the opposite angles of a quadrilateral are supplemental, it is cyclic.

2. A segment which makes equal angles with one pair of opposite sides of a cyclic quadrilateral makes equal angles with the remaining pair and with the diagonals.

3. If two opposite sides of a cyclic quadrilateral are extended to meet and a perpendicular falls on the bisector of the angle between them from the point of intersection of the diagonals, this perpendicular will bisect the angle between the diagonals.

4. If two pairs of opposite sides of a cyclic hexagon are respectively parallel to each other, the remaining pair of sides are also parallel.

5. If two circles intersect at the points A, B, and any two segments ACD, BFE are constructed through A and B, cutting one of the circles in the points C, E and the other in the points D, F, then $CE \parallel DF$.

6. If equilateral triangles are constructed on the sides of any triangle, the segments joining the vertices of the original triangle to the opposite vertices of the equilateral triangles are concurrent.

7. In the same case as #7, prove that the centers of the circles constructed about the equilateral triangles form another equilateral triangle.

8. If a quadrilateral is constructed about a circle, the angles at the center standing opposite the opposite sides are supplemental.

9. If a tangent which varies in position meets two parallel tangents, it stands opposite a right angle at the center.

10. If a hexagon is circumscribed about a circle, the sum of the angles standing opposite the center from any three alternate sides is equal to two right angles.

PROPOSITION 3.23. *UNIQUENESS OF ARCS. It is impossible to construct two similar and unequal arcs on the same side of the same chord.*

PROOF. Two similar arcs (ACB, ADB) which do not coincide cannot be constructed on the same of the same chord (AB).

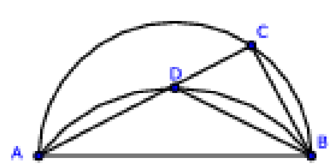

FIGURE 3.2.27. [3.23]

Suppose that we have arcs $ACB \sim ADB$ which are constructed on the same side of segment AB. Take any point D in the inner arc (ADB). Join AD, and extend it to meet the outer arc at C. Also join BC, BD. Then since the arcs are similar, $\angle ADB = \angle ACB$ [Def. 3.10], which contradicts [1.16.]. Hence, the proof. □

PROPOSITION 3.24. *SIMILAR ARCS. Similar arcs standing on equal chords are equal in length.*

PROOF. We claim that similar arc of circles (AEB, CFD) on equal chords (AB, CD) are equal in length.

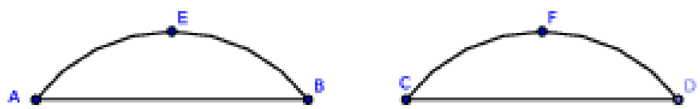

FIGURE 3.2.28. [3.24]

Since $AB = CD$, if AB is applied to CD such that the point A coincides with C and the chord AB with CD, the point B coincides with D. Because $AEB \sim CFD$, they must coincide [3.23]. Hence, the proof. □

COROLLARY. *1. Since the chords are equal in length, they are congruent; therefore the arcs, being similar, are also congruent.*

PROPOSITION 3.25. *CONSTRUCTION OF A CIRCLE FROM AN ARC. Given an arc of a circle, it is possible to construct the circle to which the arc belongs.*

PROOF. Given an arc (ABC) of a circle, we wish to construct $\circ ABC$.

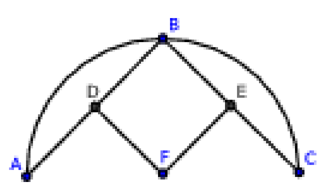

FIGURE 3.2.29. [3.25]

Take any three points A, B, C of the arc. Join AB, BC. Bisect AB at D and BC at E. Construct DF, EF at right angles to AB, BC. We claim that F, their point of intersection, is the center of the required circle.

Because DF bisects the chord AB and is perpendicular to it, the center of the circle of which ABC is an arc must lie on DF [3.1, Cor. 1]. Similarly, the center of the circle of which ABC is an arc must lie on EF. Hence the point F is the center of $\circ ABC$ (that is, the circle constructed with F as center and FA as radius). □

Propositions [3.26]-[3.29] are related in the following sense:

In [3.26], given equal angles, we must prove that we have equal arcs.

In [3.27], given equal arcs, we must prove that we have equal angles. Hence, [3.27] is the converse of [3.26], and together state that *we have equal angles if and only if we have equal arcs*.

In [3.28], given equal chords, we must prove that we have equal arcs.

In [3.29], given equal arcs, we must prove that we have equal chords. Hence, [3.29] is the converse of [3.28], and together state that *we have equal chords if and only if we have equal arcs*.

Together, these propositions essentially state that we have equal chords if and only if we have equal angles if and only if the angles stand on equal arcs.

PROPOSITION 3.26. *ANGLES AND ARCS I. In equal circles, equal angles at the centers or on the circumferences stand upon arcs of equal length.*

PROOF. In equal circles ($\circ ACB$, $\circ DFE$), equal angles at the centers ($\angle AOB$, $\angle DHE$) or at the circumferences ($\angle ACB$, $\angle DFE$) stand on equal arcs.

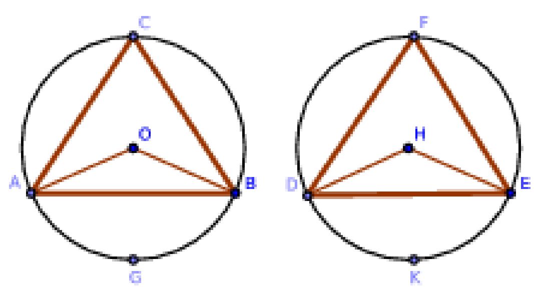

FIGURE 3.2.30. [3.26]

We prove each claim separately:

1. Suppose that $\angle AOB = \angle DHE$. We wish to show that these angles stand on equal arcs.

Because $\circ ACB$, $\circ DFE$ are equal, their radii are equal [Def. 3.1]. Therefore, the two triangles $\triangle AOB$, $\triangle DHE$ have the sides AO, OB in one respectively equal to the sides DH, HE in the other, and $\angle AOB = \angle DHE$ by hypothesis. By [1.4], $\triangle AOB \cong \triangle DHE$, and so $AB = DE$. Again, since the angles $\angle ACB$, $\angle DFE$ are the halves of the equal angles $\angle AOB$, $\angle DHE$ [3.20], $\angle ACB = \angle DFE$. By [Def. 3.10], $ACB \sim DFE$, and their chords AB, DE have been proved equal. By [3.24], the segments AB, DE are equal. And taking these equals from the circles which are equal by hypothesis, we have that the remaining arcs AGB, DKE are equal.

2. Now suppose that $\angle ACB = \angle DFE$. Since $\angle ACB = 2 \cdot \angle AOB$ and $\angle DFE = 2 \cdot \angle DHE$ [3.20], the proof follows from part 1. □

COROLLARY. *1. If the opposite angles of a cyclic quadrilateral are equal, one of its diagonals must be a diameter of the circumscribed circle.*

COROLLARY. *2. Parallel chords in a circle intercept equal arcs.*

COROLLARY. *3. If two chords intersect at any point within a circle, the sum of the opposite arcs which they intercept is equal to the arc which parallel chords intersecting on the circumference intercept. If two chords intersect at any point outside a circle, the difference of the arcs they intercept is equal to the arc which parallel chords intersecting on the circumference intercept.*

COROLLARY. *4. If two chords intersect at right angles, the sum of the opposite arcs which they intercept on the circle is a semicircle.*

PROPOSITION 3.27. *ANGLES AND ARCS II. In equal circles, angles at the centers or at the circumferences which stand on equal arcs are equal in measure.*

PROOF. In equal circles ($\circ ACB$, $\circ DFE$), angles at the centers ($\angle AOB$, $\angle DHE$) or at the circumferences ($\angle ACB$, $\angle DFE$) which stand on equal arcs (AB, DE) are equal in measure.

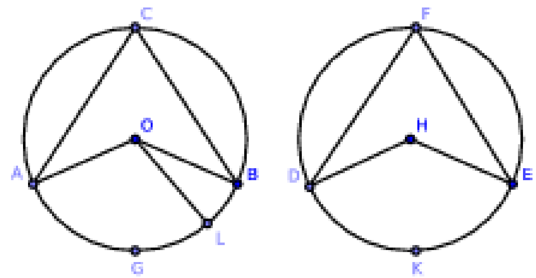

FIGURE 3.2.31. [3.27]

We prove each claim separately:

1. Consider the angles at the centers ($\angle AOB$, $\angle DHE$). Suppose that $\angle AOB > \angle DHE$ and that $\angle AOL = \angle DHE$. Since the circles are equal, the arc AL is equal to arc DE [3.26]. But $AB = DE$ by hypothesis. Hence $AB = AL$ and $AB = AL + LB$, a contradiction. A corresponding contradiction follows if we assume that $\angle AOB < \angle DHE$. Therefore, $\angle AOB = \angle DHE$.

2. Now consider the angles at the circumference. Since $\angle ACB = 2 \cdot \angle AOB$ and $\angle DFE = 2 \cdot \angle DHE$ [3.20], the proof follows from part 1. □

PROPOSITION 3.28. *CHORDS AND ARCS I. In equal circles, chords of equal length divide the circumferences into arcs such that the longer arc on the first circle is equal in length to the longer arc on the second circle, etc.*

PROOF. In equal circles ($\circ ACB$, $\circ DFE$), equal chords (AB, DE) divide the circumferences into arcs, which are respectively equal; that is, the lesser arcs are equal and the greater arcs are equal.

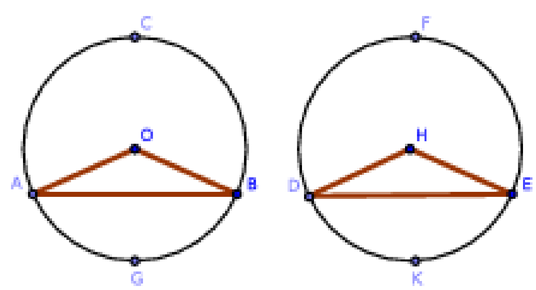

FIGURE 3.2.32. [3.28]

If the equal chords are diameters, the proof follows.

Otherwise, let O, H be the centers of $\circ ACB$, $\circ DFE$. Join AO, OB, DH, HE. Because the circles are equal, their radii are equal [Def. 3.1]. Hence the triangles $\triangle AOB$, $\triangle DHE$ have the sides AO, OB in one respectively equal to the sides DH, HE in the other, and the base AB is equal to DE by hypothesis. By [1.8], $\angle AOB = \angle DHE$, and the arc AGB is equal in length to DKE [3.26]. And since the whole circumference $AGBC$ is equal in length to the whole circumference $DKEF$, the remaining arc ACB is equal in length to the remaining arc DFE. □

PROPOSITION 3.29. *CHORDS AND ARCS II. In equal circles, equal arcs stand opposite equal chords.*

PROOF. We claim that in equal circles ($\circ ACB$, $\circ DFE$), equal arcs (AGB, DCK) stands opposite equal chords.

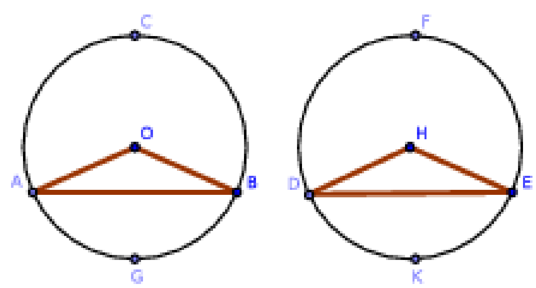

FIGURE 3.2.33. [3.29]

Let O, H be the centers of circles $\circ ACB$ and $\circ DFE$, respectively. Join AO, OB, DH, HE. Because the circles are equal, the angles $\angle AOB$, $\angle DHE$ at the centers which stand on the equal arcs AGB, DKE are themselves equal [3.27]. Again, because the triangles $\triangle AOB$, $\triangle DHE$ have the two sides AO, OB in one respectively equal to the two sides DH, HE in the other and $\angle AOB = \angle DHE$, the base AB of $\circ ACB$ is equal to the base DE of $\circ DFE$ [1.4]. Hence, the proof. □

COROLLARY. *1. Given the preceding four propositions, we have that in equal circles:*

(a) angles at the centers or at the circumferences are equal in measure if and only if they stand on arcs which are equal in length,

(b) arcs are equal in length if and only if they stand opposite chords of equal length,

(c) angles at the centers or at the circumferences are equal in measure if and only they stand on chords of equal length.

Observation: Since the two circles in the four last propositions are equal, they are congruent figures, and the truth of the propositions is made evident by superposition.

PROPOSITION 3.30. *BISECTING AN ARC. It is possible to bisect a given arc.*

PROOF. We wish to bisect the given arc ACB.

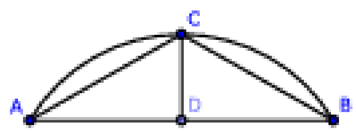

FIGURE 3.2.34. [3.30]

Construct the chord AB and bisect it at D. Construct $DC \perp AB$, intersecting the arc at C. We claim that the arc ACB is bisected at C.

Join AC, BC. Then the triangles $\triangle ADC$, $\triangle BDC$ have $AD = DB$ by construction with DC in common and the angle $\angle ADC = \angle BDC$, where each is a right angle. By [1.4], it follows that $AC = BC$. By [3.28], the arc AC is equal to the arc BC, and hence the arc $ACB = AC \oplus BC$ is bisected at C. □

Exercises.

1. Suppose that $ABCD$ is a semicircle whose diameter is AD and that the chord BC when extended meets AE (where AE is the extension of AD). Prove that if CE is equal in length to the radius, the arc $AB = 3 \cdot CD$.

2. The internal and the external bisectors of the vertical angle of a triangle inscribed in a circle meet the circumference again at points equidistant from the endpoints of the base.

3. If A is one of the points of intersection of two given circles and two chords ACD, ACD are constructed, cutting the circles in the points C, D; C', D', then the triangles $\triangle BCD$, $\triangle BC'D'$ formed by joining these to the second point B of intersection of the circles are equiangular.

4. If the vertical angle $\angle ACB$ of a triangle inscribed in a circle is bisected by a line CD which meets the circle again at D, and from D perpendiculars

DE, DF are constructed to the sides, one of which is extended, prove that $EA = BF$ and hence that $CE = \frac{1}{2}(AC + BC)$.

PROPOSITION 3.31. *ANGLES AND ARCS. In a circle,*

1. if a circle is divided into two semicircles, then the angle contained in either arc is a right angle;

2. if a circle is divided into two unequal arcs, and an angle is contained in the larger of the two arcs, then the angle contained in that arc is an acute angle;

3. if a circle is divided into two unequal arcs, and an angle is contained in the smaller of the two arcs, then the angle contained in that arc is an obtuse angle.

PROOF. In circle $\circ ABC$, we wish to show that:

1. the angle in a semicircle ($\angle ACB$) is a right angle;

2. the angle in an arc greater than a semicircle ($\angle ACD$ in arc ACD) is an acute angle;

3. the angle in an arc less than a semicircle ($\angle ACE$ in arc ACE) is an obtuse angle.

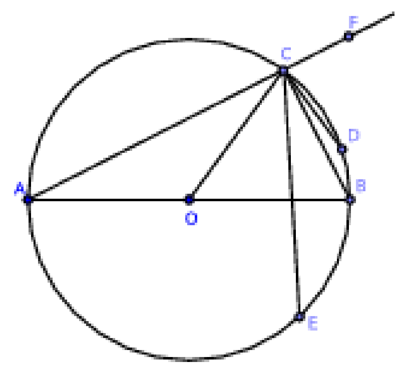

FIGURE 3.2.35. [3.31]

Construct $\circ ABC$ with center O. We prove each claim separately:

1. Let AB be the diameter of $\circ ACB$, and let C be any point on the semicircle ACB. Join AC, CB. We claim that the angle $\angle ACB$ is a right angle.

Join OC and extend AC to the ray AF. Then because $AO = OC$, $\angle ACO = \angle OAC$. Similarly, $\angle OCB = \angle CBO$. Hence,

$$\begin{aligned} \angle ACB &= \angle ACO + \angle OCB \\ &= \angle OAC + \angle CBO \\ &= \angle BAC + \angle CBA \end{aligned}$$

However by [1.32], $\angle FCB = \angle BAC + \angle CBA$ where $\angle BAC$ and $\angle CBA$ are the two interior angles of the triangle $\triangle ABC$. Hence, $\angle ACB = \angle FCB$ where each are adjacent angles, and therefore $\angle ACB$ is a right angle.

2. Consider the arc ACE which is greater than a semicircle. Join CE. Since $\angle ACB = \angle ACE + \angle BCE$, $\angle ACB > \angle ACE$. But $\angle ACB$ is a right angle by part 1 of the proof, and so $\angle ACE$ is acute.

3. Consider the arc ACD is less than a semicircle; similarly to the proof of part 2, we obtain that $\angle ACD$ is obtuse. □

COROLLARY. *1. If a parallelogram is inscribed in a circle, its diagonals intersect at the center of the circle.*

COROLLARY. *2. [3.31] holds if arcs are replaced by chords of appropriate length, mutatis mutandis.*

PROPOSITION 3.32. *TANGENT-CHORD ANGLES RELATED TO ANGLES ON THE CIRCUMFERENCE WHICH STAND ON THE CHORD. If a line is tangent to a circle, and from the point of intersection a chord is constructed cutting the circle, the angles made by this chord with the tangent are respectively equal to the angles in the alternate arcs of the circle.*

PROOF. If a line (EF) is a tangent to a circle, and from the point of intersection (A) a chord (AC) is constructed cutting the circle, we claim that the angles made by this chord with the tangent are respectively equal to the angles in the alternate arcs of the circle. We shall prove this in two cases.

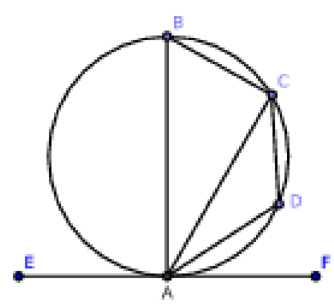

FIGURE 3.2.36. [3.32](α)

1. Construct the figures from Fig. 3.2.36(α). We wish to show that $\angle ABC = \angle FAC$.

If AC passes through the center of $\circ ABC$, then the proposition is evident because the angles are right angles.

Otherwise, from the point of intersection A construct AB such that $AB \perp EF$. Join BC. Because EF is tangent to the circle and AB is constructed from the point of intersection and is perpendicular to EF, AB passes through the center of $\circ ABC$ [3.19]. Therefore, $\angle ACB$ is right [3.31], and the sum of the two remaining angles $\angle ABC + \angle CAB$ equals one right angle. Since the angle $\angle BAF$ is right by construction, we have that $\angle ABC + \angle CAB = \angle BAF$. From this we obtain $\angle ABC = \angle BAF - \angle CAB = \angle CAF$.

2. Construct the figures from Fig. 3.2.36(β).

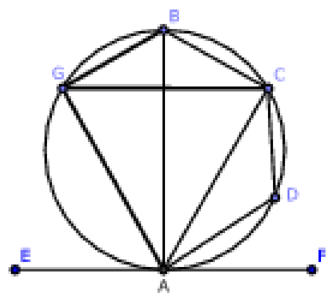

FIGURE 3.2.37. [3.32](β)

Take any point D on the arc AC. We wish to prove that $\angle CAE = \angle CDA$.

Since the quadrilateral $ABCD$ is cyclic, the sum of the opposite angles $\angle ABC + \angle CDA$ equals two right angles [3.22] and is therefore equal to the sum $\angle FAC + \angle CAE$. However, $\angle ABC = \angle FAC$ by part 1. Removing them, we obtain $\angle CDA = \angle CAE$. □

Alternatively:

PROOF. Construct the figures from Fig. 3.2.36(β). Take any point G in the semicircle AGB. Join AG, GB, GC. Then we have that $\angle AGB = \angle FAB$, since each angle is right, and $\angle CGB = \angle CAB$ [3.21]. Therefore $\angle AGC = \angle FAC$. Again, join BD, CD. Then $\angle BDA = \angle BAE$, since each angle is right, and $\angle CDB = \angle CAB$ [3.21]. Hence, $\angle CDA = \angle CAE$. □

Exercises.

1. If two circles touch, any line constructed through the point of intersection will cut off similar segments.

2. If two circles touch and any two lines are constructed through the point of intersection (cutting both circles again), the chord connecting their points of intersection with one circle is parallel to the chord connecting their points of intersection with the other circle.

3. Suppose that ACB is an arc of a circle, CE a tangent at C (meeting the chord AB extended to E), and $AD \perp AB$ where D is a point of AB. Prove that if DE be bisected at C then the arc $AC = 2 \cdot CB$.

4. If two circles touch at a point A and if ABC is a chord through A, meeting the circles at points B and C, prove that the tangents at B and C are parallel to each other, and that when one circle is within the other, the tangent at B meets the outer circle at two points equidistant from C.

5. If two circles touch externally, their common tangent at either side stands opposite a right angle at the point of intersection, and its square is equal to the rectangle contained by their diameters.

PROPOSITION 3.33. *CONSTRUCTING ARCS GIVEN A SEGMENT AND AN ANGLE. It is possible to construct an arc of a circle on a given segment such that the arc contains an angle equal to a given angle.*

PROOF. We wish to construct an arc (AB) of a circle ($\circ ABC$) on a given segment (AB) which contains an angle equal to a given angle ($\angle FGH$).

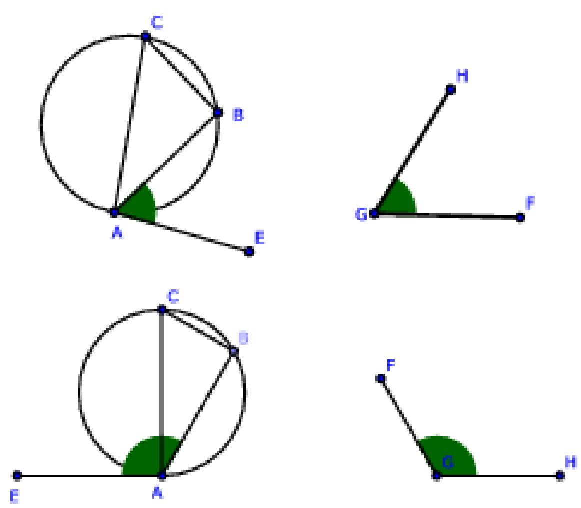

FIGURE 3.2.38. [3.33]

If $\angle FGH$ is a right angle, construct a semicircle on the given line. The proof follows since the angle in a semicircle is a right angle.

Otherwise, on the given segment AB, construct the angle $\angle BAE$ such that $\angle BAE = \angle FGH$. Construct $AC \perp AE$ and $BC \perp AB$. With AC as diameter, construct the circle $\circ CBA$. We claim that $\circ CBA$ is the required circle.

The circumference of $\circ CBA$ contains the point B (since $\angle ABC$ is right [3.31]) and also touches AE (since $\angle CAE$ is right [3.16]). Therefore, $\angle BAE$ is equal to the angle in the alternate segment [3.32]; that is, $\angle BAE = \angle ACB$. But $\angle BAE = \angle FGH$ by construction, and so $\angle FGH = \angle ACB$, and the arc AB contains the angle $\angle ACB$. □

Exercises.

1. Construct a triangle, being given the base, vertical angle, and any of the following data:

(a) a perpendicular.

(b) the sum or difference of the sides.

(c) the sum or difference of the squares of the sides.

(d) the side of the inscribed square on the base.

(e) the median that bisects the base.

2. If lines are constructed from a fixed point to all the points of the circumference of a given circle, prove that the locus of all their points of bisection is a circle.

3. Given the base and vertical angle of a triangle, find the locus of the midpoint of the line joining the vertices of equilateral triangles constructed on the sides.

4. In the same case, find the loci of the angular points of a square constructed on one of the sides.

PROPOSITION 3.34. *DIVIDING ARCS GIVEN AN ANGLE. It is possible to divide the circumference of a circle such that the separated arc contains an angle equal to a given angle.*

PROOF. From a given circle ($\circ BCA$), we wish to cut off an arc which contains an angle equal to a given angle ($\angle FGH$).

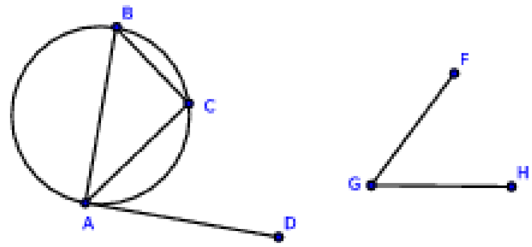

FIGURE 3.2.39. [3.34]

Take any point A on the circumference and construct the tangent AD. Construct the angle $\angle DAC$ such that $\angle DAC = \angle FGH$. Take any point B on the alternate arc. Join BA, BC. Then $\angle DAC = \angle ABC$ [3.32]. But $\angle DAC = \angle FGH$ by construction, and so $\angle ABC = \angle FGH$. □

PROPOSITION 3.35. *AREAS OF RECTANGLES CONSTRUCTED ON CHORDS. If two chords of a circle intersect at one and only one point within the circle, then the area of the rectangle contained by the divided segments of the first chord is equal in area to the rectangle contained by the divided segments of the second chord [Def. 2.4].*

PROOF. If two chords (AB, CD) of a circle ($\circ ACB$) intersect at a point (E) within the circle, the rectangles ($AE.EB$, $CE.ED$) contained by the segments are equal in area.

We prove this proposition in four cases:

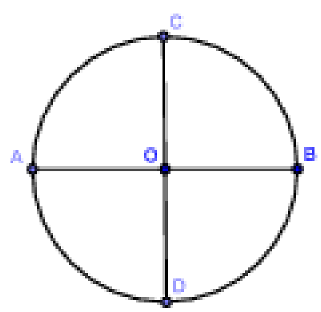

FIGURE 3.2.40. [3.35], case 1

1. If the point of intersection is the center of $\circ ACB$, each rectangle is equal in area to the square of the radius. Hence, the proof.

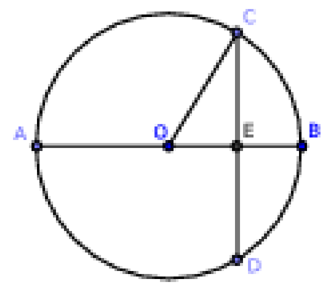

FIGURE 3.2.41. [3.35], case 2

2. Suppose that the chord AB passes through the center of $\circ ABC$ and that the chord CD does not. Further suppose that $AB \perp CD$.

Join OC. Because AB passes through the center and cuts the other chord CD which does not pass through the center at right angles, AB bisects CD [3.3]. Because AB is divided equally at O and unequally at E, by [2.5], we have that $AE.EB+OE^2 = OB^2$. Since $OB = OC$, we also have that $AE.EB+OE^2 = OC^2$. But $OC^2 = OE^2 + EC^2$ [1.47]: therefore $AE.EB + OE^2 = OE^2 + EC^2$. Subtracting OE^2 from both sides of the equality, we obtain $AE.EB = EC^2$. But $EC^2 = CE.ED$ since $CE = ED$. Therefore, $AE.EB = CE.ED$.

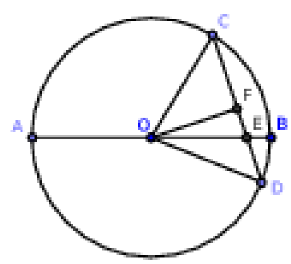

FIGURE 3.2.42. [3.35], case 3

3. Let O be the center of $\circ ACB$, and let AB pass through the center of $\circ ACB$ and cut CD such that $AB \not\perp CD$.

Construct $OF \perp CD$ [1.11]. Join OC, OD. Since CD is cut at right angles by OF and OF passes through O, CD is bisected at F [3.3] and divided unequally at E. Hence by [2.5], $CE.ED + FE^2 = FD^2$. Adding OF^2 to each side of the equality, we obtain:

$$\begin{aligned} CE.ED + FE^2 + OF^2 &= FD^2 + OF^2 \\ CE.ED + OE^2 &= OD^2 \\ CE.ED + OE^2 &= OB^2 \end{aligned}$$

Again, since AB is bisected at O and divided unequally at E, $AE.EB + OE^2 = OB^2$ [2.5]. It follows that

$$\begin{aligned} CE.ED + OE^2 &= AE.EB + OE^2 \\ CE.ED &= AE.EB \end{aligned}$$

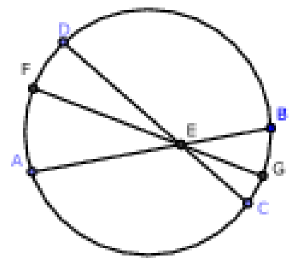

FIGURE 3.2.43. [3.35], case 4

4. Suppose neither chord passes through the center. Through E, their point of intersection, construct the diameter FG. By case 3, the rectangle $FE.EG = AE.EB$ and $FE.EG = CE.ED$. Hence, $AE.EB = CE.ED$. □

COROLLARY. *1. If a chord of a circle is divided at any point within the circle, the rectangle contained by its segments is equal to the difference between the square of the radius and the square of the segment constructed from the center to the point of section.*

COROLLARY. *2. If the rectangle contained by the segments of one of two intersecting segments is equal to the rectangle contained by the segments of the other, the four endpoints are concyclic.*

COROLLARY. *3. If two triangles are equiangular, the rectangle contained by the non-corresponding sides about any two equal angles are equal.*

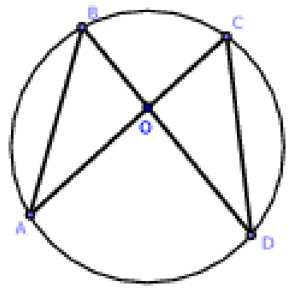

FIGURE 3.2.44. [3.35], Cor. 3

Let $\triangle ABO$, $\triangle DCO$ be the equiangular triangles, and let them be placed so that the equal angles at O are vertically opposite and that the non-corresponding sides AO, CO stand in one segment. Then the non-corresponding sides BO, OD form the segment BD. Since $\angle ABD = \angle ACD$, the points A, B, C, D are concyclic [3.21, Cor. 1]. Hence, $AO.OC = BO.OD$ [3.35].

Exercises.

1. In any triangle, the rectangle contained by two sides is equal in area to the rectangle contained by the perpendicular on the third side and the diameter of the circumscribed circle.

Definition: The supplement of an arc is the difference between the arc and a semicircle.

2. The rectangle contained by the chord of an arc and the chord of its supplement is equal to the rectangle contained by the radius and the chord of twice the supplement.

3. If the base of a triangle is given with the sum of the sides, the rectangle contained by the perpendiculars from the endpoints of the base on the external bisector of the vertical angle is given.

4. If the base and the difference of the sides is given, the rectangle contained by the perpendiculars from the endpoints of the base on the internal bisector is given.

5. Through one of the points of intersection of two circles, construct a secant so that the rectangle contained by the intercepted chords may be given, or is a maximum.

6. If the sum of two arcs AC, CB of a circle is less than a semicircle, the rectangle $AC.CB$ contained by their chords is equal to the rectangle contained by the radius and the excess of the chord of the supplement of their difference above the chord of the supplement of their sum.

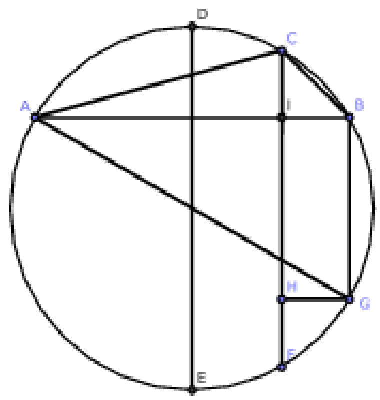

FIGURE 3.2.45. [5.35, #6]

Construct the diameter DE such that $DE \perp AB$, and construct the chords CF, BG parallel to DE. It is evident that the difference between the arcs AC, CB is equal to $2 \cdot CD$, and therefore equals $CD + EF$. Hence the arc CBF is the supplement of the difference and CF is the chord of that supplement. Again, since the angle $\angle ABG$ is right, the arc ABG is a semicircle. Hence BG is the supplement of the sum of the arcs AC, CB, and therefore the segment BG is the chord of the supplement of the sum. By #1, the rectangle $AC.CB$ is equal to the rectangle contained by the diameter and CI, and therefore equal to the rectangle contained by the radius and $2 \cdot CI$. But the difference between CF and BG is evidently equal to $2 \cdot CI$. Hence the rectangle $AC.CB$ is equal to the rectangle contained by the radius and the difference between the chords CF, BG.

7. If we join AF, BF, we find that the rectangle $AF.FB$ is equal in area to the rectangle contained by the radius and $2 \cdot FI$; that is, it is equal to the rectangle contained by the radius and the sum of CF and BG. Hence, if the sum of two arcs of a circle is greater than a semicircle, the rectangle contained by their chords is equal to the rectangle contained by the radius and the sum of the chords of the supplements of their sum and their difference.

8. Through a given point, construct a transversal cutting two given lines so that the rectangle contained by the segments intercepted between it and the line may be given.

PROPOSITION 3.36. *THE AREA OF RECTANGLES CONSTRUCTED ON A TANGENT AND A POINT OUTSIDE THE CIRCLE I. Suppose we are given a circle and a point outside of the circle. If two segments are constructed from the point to the circle, the first of which intersects the circle at two points and the* *second of which is tangent to the circle, then the area of the rectangle contained by the subsegments of the first segment is equal to the square on the tangent.*

PROOF. If from any point (P) outside of the circle $\circ ATB$ two segments are constructed to meet P, one of which (PT) is a tangent and the other (PA) a secant, then the rectangle contained by the segments of the secant ($AP.BP$) is equal in area to the square of the tangent (PT); or, $AP.BP = PT^2$.

We solve the proposition in two cases:

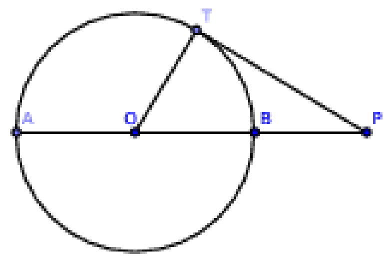

FIGURE 3.2.46. [3.36], case 1

1. Let PA pass through the center O of $\circ ATB$. Join OT. Because AB is bisected at O and divided externally at P, the rectangle $AP.BP + OB^2 = OP^2$ [2.6]. Since PT is a tangent to $\circ ATB$ and OT is constructed from the center to the point of intersection, the angle $\angle OTP$ is right [3.18]. Hence $OT^2 + PT^2 = OP^2$.

Therefore $AP.BP + OB^2 = OT^2 + PT^2$. But $OB^2 = OT^2$, and so $AP.BP = PT^2$.

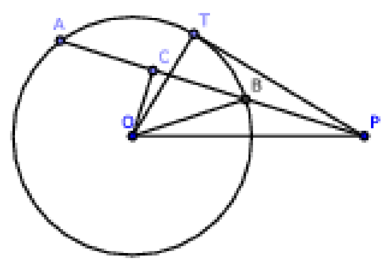

FIGURE 3.2.47. [3.36], case 2

2. If AB does not pass through the center O, construct the perpendicular OC on AB. Join OT, OB, OP. Because OC, a segment through the center, cuts AB, which does not pass through the center at right angles, OC bisects AB [3.3]. Since AB is bisected at C and divided externally at P, we have that $AP.BP + CB^2 = CP^2$ [2.6]. Adding OC^2 to each side, we obtain:

$$AP.BP + CB^2 + OC^2 = CP^2 + OC^2$$
$$AP.BP + OB^2 = OP^2$$

We also have that $OT^2 + PT^2 = OP^2$, from which it follows that $AP.BP + OB^2 = OT^2 + PT^2$. Subtracting OB^2 and OT^2 (since $OB = OT$), we have that $AP.BP = PT^2$. □

Note: The two propositions [3.35] and [3.36] may be written as one statement: the rectangle $AP.BP$ contained by the segments of any chord of a given circle passing through a fixed point P, either within or outside of the circle, is constant.

Suppose O is the center the circle, and join OA, OB, OP. Then $\triangle OAB$ is an isosceles triangle, and OP is a segment constructed from its vertex to a point P in the base or the extended base. It follows that the rectangle $AP.BP$ is equal to the difference of the squares of OB and OP; therefore, it is constant.

COROLLARY. *1. If two segment AB, CD are extended to meet at P, and if the rectangle $AP.BP = CP.DP$, then the points A, B, C, D are concyclic (compare p3.35, Cor. 2]).*

COROLLARY. *2. Tangents to two circles from any point in their common chord are equal (compare [3.17, #6]).*

COROLLARY. *3. The common chords of any three intersecting circles are concurrent (compare [3.17, #7]).*

Exercise.

1. If from the vertex A of $\triangle ABC$, the segment AD is constructed which meets CB extended to D and creates the angle $\angle BAD = \angle ACB$, prove that $DB.DC = DA^2$.

PROPOSITION 3.37. *THE AREA OF RECTANGLES CONSTRUCTED ON A TANGENT AND A POINT OUTSIDE THE CIRCLE II. Suppose we are given a circle and a point outside of the circle. If two segments are constructed from the point to the circle, the first of which intersects the circle at two points, and the area of the rectangle contained by the subsegments of the first segment is equal to the square on the second segment, then the second segment is tangent to the circle.*

PROOF. If the rectangle ($AP.BP$) contained by the segments of a secant and constructed from any point (P) outside of the circle ($\circ ATB$) is equal in area to the square on the segment (PT) constructed from the same point to meet the circle, then the segment which meets the circle is a tangent to that circle.

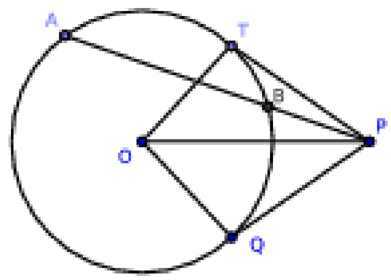

FIGURE 3.2.48. [3.37]

From P, construct PQ touching the circle [3.17]. Let O be the center of $\circ ATQ$ and join OP, OQ, OT. By hypothesis, $AP.BP = PT^2$; by [3.36], $AP.BP = PQ^2$. Hence $PT^2 = PQ^2$, and so $PT = PQ$.

Consider the triangles $\triangle OTP$, $\triangle OQP$. Each have $OT = OQ$, $TP = QP$, and the base OP in common. By [1.8], $\triangle OTP \cong \triangle OQP$, and so $\angle OTP = \angle OQP$. But $\angle OQP$ is a right angle since PQ is a tangent [3.38]; hence $\angle OTP$ is right, and therefore PT is a tangent to $\circ ATB$ [3.16]. □

COROLLARY. *1. Suppose we are given a circle and a point outside of the circle where two segments are constructed from the point to the circle, the first of which intersects the circle at two points. Then the second segment is tangent to the circle if and only if the area of the rectangle contained by the subsegments of the first segment is equal to the square on the tangent.*

Exercises.

1. Construct a circle passing through two given points and fulfilling either of the following conditions:

(a) touching a given line;

(b) touching a given circle.

2. Construct a circle through a given point and touching two given lines; or touching a given line and a given circle.

3. Construct a circle passing through a given point having its center on a given line and touching a given circle.

4. Construct a circle through two given points and intercepting a given arc on a given circle.

5. If A, B, C, D are four collinear points and EF is a common tangent to the circles constructed upon AB, CD as diameters, then prove that the triangles $\triangle AEB$, $\triangle CFD$ are equiangular.

6. The diameter of the circle inscribed in a right-angled triangle is equal to half the sum of the diameters of the circles touching the hypotenuse, the perpendicular from the right angle of the hypotenuse, and the circle constructed about the right-angled triangle.

Examination questions on chapter 3

1. What is the subject-matter of chapter 3?

2. Define equal circles.

3. Define a chord.

4. When does a secant become a tangent?

5. What is the difference between an arc and a sector?

6. What is meant by an angle in a segment?

7. If an arc of a circle is one-sixth of the whole circumference, what is the magnitude of the angle in it?

8. What are segments?

9. What is meant by an angle standing on a segment?

10. What are concyclic points?

11. What is a cyclic quadrilateral?

12. How many intersections can a line and a circle have?

13. How many points of intersection can two circles have?

14. Why is it that if two circles touch they cannot have any other common point?

15. State a proposition that encompasses [3.11] and [3.12].

16. What proposition is #16 a limiting case of?

17. What is the modern definition of an angle?

18. How does the modern definition of an angle differ from Euclid's?

19. State the relations between [3.16], [3.18] and [3.19].

20. What propositions are [3.16], [3.18] and [3.19] limiting cases of?

21. How many common tangents can two circles have?

22. What is the magnitude of the rectangle of the segments of a chord constructed through a point 3.65m distant from the center of a circle whose radius is 4.25m?

23. The radii of two circles are 4.25 and 1.75 ft respectively, and the distance between their centers 6.5 ft. Find the lengths of their direct and their transverse common tangents.

24. If a point is h feet outside the circumference of a circle whose diameter is 7920 miles, prove that the length of the tangent constructed from it to the circle is $\sqrt{3h/2}$ miles.

25. Two parallel chords of a circle are 12 inches and 16 inches respectively and the distance between them is 2 inches. Find the length of the diameter.

26. What is the locus of the centers of all circles touching a given circle in a given point?

27. What is the condition that must be fulfilled that four points may be concyclic?

28. If the angle in a segment of a circle equals 1.5 right angles, what part of the whole circumference is it?

29. Mention the converse propositions of chapter 3 which are proved directly.

30. What is the locus of the midpoints of equal chords in a circle?

31. The radii of two circles are 6 and 8, and the distance between their centers is 10. Find the length of their common chord.

32. If a figure of any even number of sides is inscribed in a circle, prove that the sum of one set of alternate angles is equal to the sum of the remaining angles.

Chapter 3 exercises.

1. If two chords of a circle intersect at right angles, the sum of the squares on their segments is equal to the square on the diameter.

2. If a chord of a given circle stands opposite a right angle at a fixed point, the rectangle of the perpendiculars on it from the fixed point and from the center of the given circle is constant. Also, the sum of the squares of perpendiculars on it from two other fixed points (which may be found) is constant.

3. If through either of the points of intersection of two equal circles any line is constructed meeting them again in two points, these points are equally distant from the other intersection of the circles.

4. Construct a tangent to a given circle so that the triangle formed by it and two fixed tangents to the circle shall be:

(a) a maximum;

(b) a minimum.

5. If through the points of intersection A, B of two circles any two segments ACD, BEF are constructed parallel to each other which meet the circles again at C, D, E, F, then we have that $CD = EF$.

6. In every triangle, the bisector of the greatest angle is the least of the three bisectors of the angles.

7. The circles whose diameters are the four sides of any cyclic quadrilateral intersect again in four concyclic points.

8. The four angular points of a cyclic quadrilateral determine four triangles whose orthocenters (the intersections of their perpendiculars) form an equal quadrilateral.

9. If through one of the points of intersection of two circles we construct two common chords, the segments joining the endpoints of these chords make a given angle with each other.

10. The square on the perpendicular from any point in the circumference of a circle on the chord of contact of two tangents is equal to the rectangle of the perpendiculars from the same point on the tangents.

11. Find a point on the circumference of a given circle such that the sum of the squares on whose distances from two given points is either a maximum or a minimum.

12. Four circles are constructed on the sides of a quadrilateral as diameters. Prove that the common chord of any two on adjacent sides is parallel to the common chord of the remaining two.

13. The rectangle contained by the perpendiculars from any point in a circle on the diagonals of an inscribed quadrilateral is equal to the rectangle contained by the perpendiculars from the same point on either pair of opposite sides.

14. The rectangle contained by the sides of a triangle is greater than the square on the internal bisector of the vertical angle by the rectangle contained by the segments of the base.

15. If through A, one of the points of intersection of two circles, we construct any line ABC which cuts the circles again at B and C, the tangents at B and C intersect at a given angle.

16. If a chord of a given circle passes through a given point, the locus of the intersection of tangents at its endpoints is a straight line.

17. The rectangle contained by the distances of the point where the internal bisector of the vertical angle meets the base and the point where the perpendicular from the vertex meets it from the midpoint of the base is equal to the square on half the difference of the sides.

18. State and prove the proposition analogous to [3.17] for the external bisector of the vertical angle.

19. The square on the external diagonal of a cyclic quadrilateral is equal to the sum of the squares on the tangents from its endpoints to the circumscribed circle.

20. If a "movable" circle touches a given circle and a given line, the chord of contact passes through a given point.

21. If A, B, C are three points in the circumference of a circle, and D, E are the midpoints of the arcs AB, AC, and if the segment DE intersects the chords AB, AC at F and G, then $AF = AG$.

22. If a cyclic quadrilateral is such that a circle can be inscribed in it, the lines joining the points of contact are perpendicular to each other.

23. If through the point of intersection of the diagonals of a cyclic quadrilateral the minimum chord is constructed, that point will bisect the part of the chord between the opposite sides of the quadrilateral.

24. Given the base of a triangle, the vertical angle, and either the internal or the external bisector at the vertical angle, construct the triangle.

25. If through the midpoint A of a given arc BAC we construct any chord AD, cutting BC at E, then the rectangle $AD.AE$ is constant.

26. The four circles circumscribing the four triangles formed by any four lines pass through a common point.

27. If X, Y, Z are any three points on the three sides of a triangle $\triangle ABC$, the three circles about the triangles $\triangle YAZ$, $\triangle ZBX$, $\triangle XCY$ pass through a common point.

28. If the position of the common point in the previous exercise are given, the three angles of the triangle $\triangle XYZ$ are given, and conversely.

29. Place a given triangle so that its three sides shall pass through three given points.

30. Place a given triangle so that its three vertices shall lie on three given lines.

31. Construct the largest triangle equiangular to a given one whose sides shall pass through three given points.

32. Construct the smallest possible triangle equiangular to a given one whose vertices shall lie on three given lines.

33. Construct the largest possible triangle equiangular to a given triangle whose sides shall touch three given circles.

34. If two sides of a given triangle pass through fixed points, the third touches a fixed circle.

35. If two sides of a given triangle touch fixed circles, the third touches a fixed circle.

36. Construct an equilateral triangle having its vertex at a given point and the endpoints of its base on a given circle.

37. Construct an equilateral triangle having its vertex at a given point and the endpoints of its base on two given circles.

38. Place a given triangle so that its three sides touch three given circles.

39. Circumscribe a square about a given quadrilateral.

40. Inscribe a square in a given quadrilateral.

41. Construct the following circles:

(a) orthogonal (cutting at right angles) to a given circle and passing through two given points;

(b) orthogonal to two others, and passing through a given point;

(c) orthogonal to three others.

42. If from the endpoints of a diameter AB of a semicircle two chords AD, BE are constructed which meet at C, we have that $AC.AD + BC.BE = AB^2$.

43. If $ABCD$ is a cyclic quadrilateral, and if we construct any circle passing through the points A and B, another through B and C, a third through C and D, and a fourth through D and A, then these circles intersect successively at four other points E, F, G, H, forming another cyclic quadrilateral.

44. If $\triangle ABC$ is an equilateral triangle, what is the locus of the point M, if $MA = MB + MC$?

45. In a triangle, given the sum or the difference of two sides and the angle formed by these sides both in magnitude and position, the locus of the center of the circumscribed circle is a straight line.

46. Construct a circle:

(a) through two given points which bisect the circumference of a given circle;

(b) through one given point which bisects the circumference of two given circles.

47. Find the locus of the center of a circle which bisects the circumferences of two given circles.

48. Construct a circle which bisects the circumferences of three given circles.

49. If CD is a perpendicular from any point C in a semicircle on the diameter AB, $\circ EFG$ is a circle touching DB at E, CD at F, and the semicircle at G, then prove that:

(a) the points A, F, G are collinear;

(b) $AC = AE$.

50. Being given an obtuse-angled triangle, construct from the obtuse angle to the opposite side a segment whose square is equal to the rectangle contained by the segments into which it divides the opposite side.

51. If O is a point outside a circle whose center is E and two perpendicular segments passing through O intercept chords AB, CD on the circle, then prove that $AB^2 + CD^2 + 4OE^2 = 8R^2$.

52. The sum of the squares on the sides of a triangle is equal to twice the sum of the rectangles contained by each perpendicular and the portion of it comprised between the corresponding vertex and the orthocenter. It is also equal to $12R^2$ minus the sum of the squares of the distances of the orthocenter from the vertices.

53. If two circles touch at C, if D is any point outside the circles at which their radii through C stands opposite equal angles, and if DE, DF are tangent from D, prove that $DE.DF = DC^2$.

CHAPTER 4

Inscription and Circumscription

This chapter contains sixteen propositions, four of which relate to triangles, four to squares, four to pentagons, and four to miscellaneous figures.

4.1. Definitions

1. If two polygons are related such that the angular points of one lie on the sides of the other, then:

(a) the former is said to be *inscribed* in the latter;

(b) the latter is said to be *circumscribed* around or about the former.

2. A polygon is said to be *inscribed* in a circle when its angular points are on the circumference. Reciprocally, a polygon is said to be *circumscribed* about or around a circle when each side touches the circle.

3. A circle is said to be *inscribed* in a polygon when it touches each side of the figure. Reciprocally, a circle is said to be *circumscribed* about or around a polygon when it passes through each angular point of the figure.

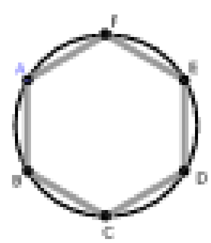

FIGURE 4.1.1. [Def. 4.1] The hexagon $ABCDEF$ is inscribed in the circle $\circ ABC$; the circle $\circ ABC$ is circumscribed about the hexagon $ABCDEF$.

4. A polygon which is both equilateral and equiangular is said to be *regular*.

4.2. Propositions from Book IV

PROPOSITION 4.1. *CONSTRUCTING A CHORD INSIDE A CIRCLE. In a given circle, it is possible to construct a chord equal in length to a given segment not greater than the circle's diameter.*

PROOF. Given with diameter AC, we wish to construct a chord in $\circ ABC$ equal in length to a given segment, DG (where $AC \geq DG$).

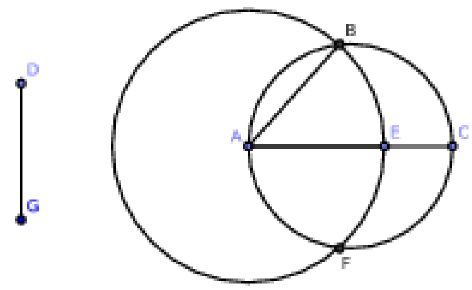

FIGURE 4.2.1. [4.1]

If $AC = DG$, then the required chord already exists within the circle (i.e., its diameter).

Otherwise, cut subsegment AE from diameter AC such that $AE = DG$ [1.3]. With A as center and AE as radius, construct the circle $\circ EBF$, cutting the circle $\circ ABC$ at the points B, F. Join AB. We claim that AB is the required chord.

Because A is the center of $\circ EBF$, $AB = AE$. But $AE = DG$ by construction, and so $AB = DG$. Since AB is a chord of $\circ ABC$, the proof follows. □

PROPOSITION 4.2. *INSCRIBE A TRIANGLE INSIDE A CIRCLE. In a given circle, it is possible to inscribe a triangle equiangular to a given triangle.*

PROOF. We wish to inscribe a triangle equiangular to a given triangle ($\triangle DEF$) in a given circle ($\circ ABC$).

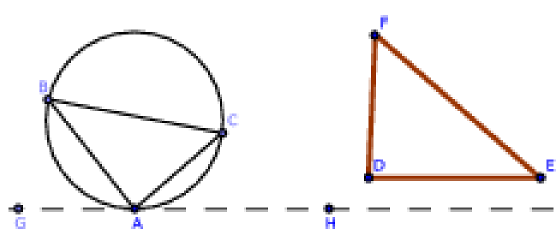

FIGURE 4.2.2. [4.2]

At a point A on the circumference of $\circ ABC$, construct the tangent line GAH. Construct $\angle HAC = \angle DEF$ and $\angle GAB = \angle DFE$ [1.23]. Join BC. We claim that $\triangle ABC$ fulfills the required conditions.

Since $\angle DEF = \angle HAC$ by construction and $\angle HAC = \angle ABC$ in the alternate segment [3.32], we have that $\angle DEF = \angle ABC$. Similarly, $\angle DFE = \angle ACB$. By [1.32], $\angle FDE = \angle BAC$. Hence the triangle $\triangle ABC$ inscribed in $\circ ABC$ is equiangular to $\triangle DEF$. □

PROPOSITION 4.3. *CIRCUMSCRIBE A TRIANGLE ABOUT A CIRCLE. It is possible to circumscribe a triangle about a circle such that the triangle is equiangular to a given triangle.*

PROOF. We wish to to construct a triangle equiangular to a given triangle ($\triangle DEF$) about a given circle ($\circ ABC$).

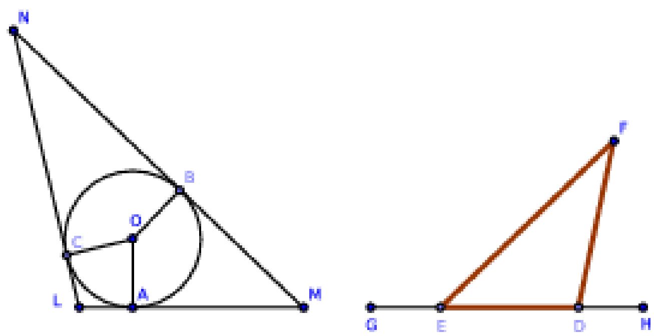

FIGURE 4.2.3. [4.3]

Extend side DE of $\triangle DEF$ to the segment GH, and from the center O of $\circ BCA$ construct any radius OA. Construct $\angle AOB = \angle GEF$ and $\angle AOC = \angle HDF$ [1.23]. At the points A, B, and C, construct the tangents LM, MN, NL to $\circ BCA$. We claim that $\triangle LMN$ fulfills the required conditions.

Because AM touches $\circ BCA$ at A, the angle $\angle OAM$ is right [3.18]. Similarly, the angle $\angle MBO$ is right; but the sum of the four angles of the quadrilateral $OAMB$ is equal to four right angles [1.32, Cor.3]. Therefore the sum of the two remaining angles $\angle AOB + \angle AMB$ is two right angles. By [1.13], $\angle GEF + \angle FED$ is two right angles [1.13], and so $\angle AOB + \angle AMB = \angle GEF + \angle FED$. But $\angle AOB = \angle GEF$ by construction; hence $\angle AMB = \angle FED$; similarly, $\angle ALC = \angle EDF$. Therefore by [1.32], $\angle BNC = \angle DFE$, and the triangle $\triangle LMN$ is equiangular to $\triangle DEF$. □

PROPOSITION 4.4. *INSCRIBE A CIRCLE IN A TRIANGLE. It is possible to inscribe a circle in a given triangle.*

PROOF. We wish to inscribe a circle ($\circ DEF$) in a given triangle ($\triangle ABC$).

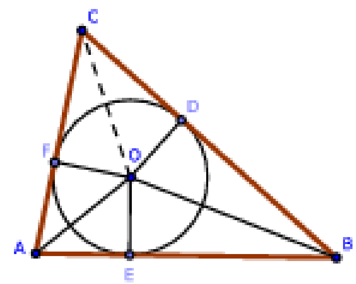

FIGURE 4.2.4. [4.4]

Bisect angles $\angle CAB$, $\angle ABC$ of $\triangle ABC$ by the segments AO, BO; we claim that O, their point of intersection, is the center of the required circle.

From O construct the perpendicular segments OD, OE, OF on the sides of $\triangle ABC$. Consider the triangles $\triangle OAE$ and $\triangle OAF$: $\angle OAE = \angle OAF$ by construction; $\angle AEO = \angle AFO$ because each is right; and the side OA stands in common. Hence, $OE = OF$ [1.26]. Similarly, $OD = OF$. Therefore $OD = OE = OF$, and by [3.9], the circle constructed with O as center and OD as radius will pass through the points D, E, F by construction. Since each of the angles $\angle ODB$, $\angle OEA$, $\angle OFA$ is right, each touches the respective sides of the triangle $\triangle ABC$ [3.16]. Therefore, the circle $\circ DEF$ is inscribed in the triangle $\triangle ABC$. □

Definition: The bisectors of the three internal angles of a triangle are concurrent. Their point of intersection is called the **incenter** of the triangle.

Exercises.

1. In [4.4]: if the points O, C are joined, prove that the angle $\angle ACB$ is bisected. Hence, we prove the existence of the **incenter** of a triangle.

2. If the sides BC, CA, AB of the triangle $\triangle ABC$ are denoted by a, b, c, and half their sum is denoted by s, the distances of the vertices A, B, C of the triangle from the points of contact of the inscribed circle are respectively $s-a$, $s-b$, $s-c$.

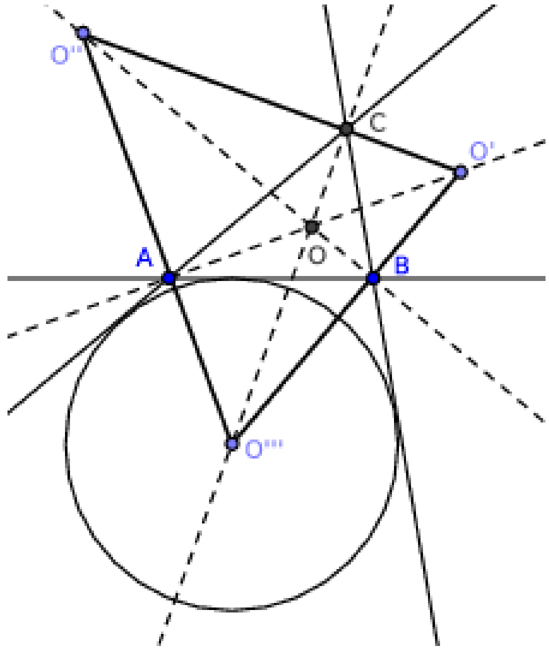

FIGURE 4.2.5. [4.4, #3]

3. If the external angles of the triangle $\triangle ABC$ are bisected as in Fig. 4.2.5, the three angular points O', O'', O''' of the triangle formed by the three bisectors are the centers of three circles, each touching one side externally and the other two when extended. These three circles are called the *escribed circles* of the triangle $\triangle ABC$.

4. The distances of the vertices A, B, C from the points of contact of the escribed circle which touches AB externally are $s{-}b$, $s{-}a$, s.

5. The center of the inscribed circle, the center of each escribed circle, and two of the angular points of the triangle are concyclic. Also, any two of the escribed centers are concyclic with the corresponding two of the angular points of the triangle.

6. Of the four points O, O', O'', O''', any one is the orthocenter of the triangle formed by the remaining three.

7. The three triangles $\triangle BCO$, $\triangle CAO$,$\triangle ABO$ are equiangular.

8. Confirm that $CO.CO = ab$, $AO.AO = bc$, $BO.BO = ca$.

9. Since the whole triangle $\triangle ABC$ is made up of the three triangles $\triangle AOB$, $\triangle BOC$, $\triangle COA$, we see that the rectangle contained by the sum of the three sides and the radius of the inscribed circle is equal to twice the area of the triangle. Hence, if r denotes the radius of the inscribed circle, rs =area of $\triangle ABC$.

10. If r' denotes the radius of the escribed circle which touches the side a externally, it may be shown that $r'(s{-}a) =$area of the triangle.

11.Show that $rr' = s{-}b.s{-}c$.

12. Show that the square of the area $= s.(s{-}a).(s{-}b).(s{-}c)$.

13. Show that the square of the area = $r.r'.r''.r'''$.

14. If $\triangle ABC$ is a right triangle where the angle at C is right, then $r = s{-}c$, $r' = s{-}b$, $r'' = s{-}a$, and $r''' = s$.

15. Given the base of a triangle, the vertical angle, and the radius of the inscribed or any of the escribed circles, construct it.

PROPOSITION 4.5. *CIRCUMSCRIBE A CIRCLE ABOUT A TRIANGLE. It is possible to circumscribe a circle about a given triangle.*

PROOF. We wish to construct a circle ($\circ ABC$) about a given triangle ($\triangle ABC$).

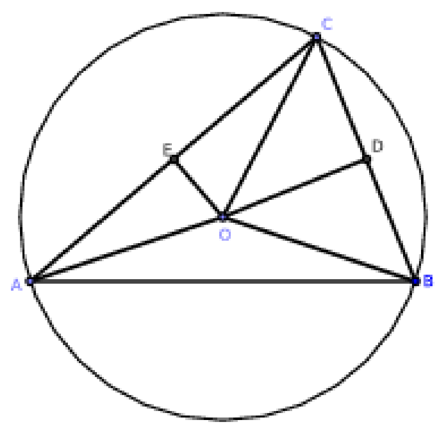

FIGURE 4.2.6. [4.5]

Bisect sides BC, AC of $\triangle ABC$ at the points D, E, respectively. Construct DO, EO at right angles to BC, CA. We claim that O, the point of intersection of the perpendiculars, is the center of the required circle $\circ ABC$.

Join OA, OB, OC. Consider the triangles $\triangle BDO$, $\triangle CDO$: they have the side $BD = CD$ by construction, side DO in common, and $\angle BDO = \angle CDO$ because each is right. By [1.4], $\triangle BDO \cong \triangle CDO$, and so $BO = OC$. Similarly, $AO = OC$. Therefore, $AO = BO = CO$, and by [3.9], the circle $\circ ABC$ constructed with O as center and OA as radius will pass through the points A, B, and C; thus, $\circ ABC$ is circumscribed about the triangle $\triangle ABC$. $\square$

COROLLARY. *1. Since the perpendicular from O to AB bisects AB [3.3], we see that the perpendiculars at the midpoints of the sides of a triangle are concurrent. (See the following Definition.)*

Definition: The circle $\circ ABC$ is called the circumcircle, its radius the circumradius, and its center the **circumcenter** of the triangle.

Exercises.

1. The three altitudes of a triangle ($\triangle ABC$) are concurrent. (This proves the existence of the **orthocenter** of a circle.)

2. Prove that the three rectangles $OA.OP$, $OB.OQ$, $OC.OR$ are equal in area.

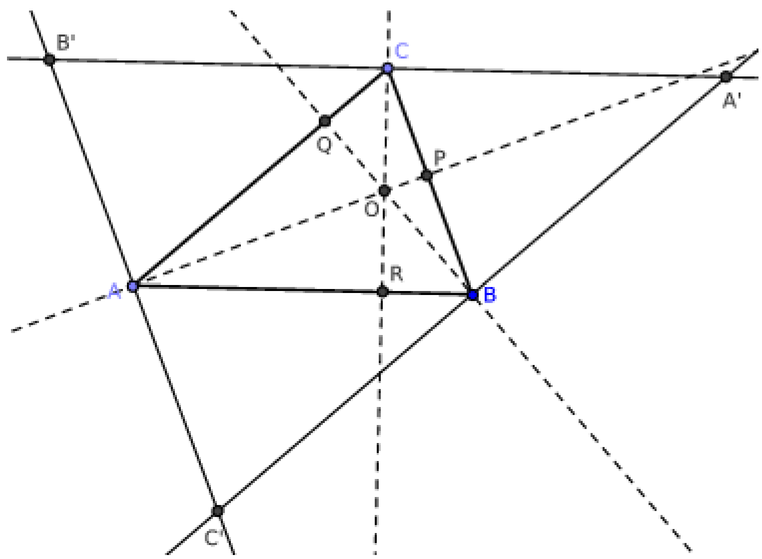

FIGURE 4.2.7. [4.5, #2]

Definition: The circle with O as center, the square of whose radius is equal $OA.OP = OB.OQ = OC.OR$, is called the *polar circle* of the triangle $\triangle ABC$.

3. If the altitudes of a triangle are extended to meet a circumscribed circle, the intercepts between the orthocenter and the circle are bisected by the sides of the triangle.

Definition: The nine-points circle is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant concyclic points defined from the triangle. These nine points are:

(a) the midpoint of each side of the triangle

(b) the foot of each altitude

(c) the midpoint of the line segment from each vertex of the triangle to the orthocenter (where the three altitudes meet; these line segments lie on their respective altitudes). [1] See Fig. 4.2.9.

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FIGURE 4.2.8. [4.5, #4] The nine-points circle

4. The circumcircle of a triangle is the "nine points circle" of each of the four triangles formed by joining the centers of the inscribed and escribed circles.

5. The distances between the vertices of a triangle and its orthocenter are respectively the doubles of the perpendiculars from the circumcenter on the sides.

6. The radius of the "nine points circle" of a triangle is equal to half its circumradius.

Note: the orthocenter, centroid, and circumcenter of any triangle are collinear; the line they lie on is named the Euler line[2].

PROPOSITION 4.6. *INSCRIBE A SQUARE IN A CIRCLE. It is possible to to inscribe a square in a given circle.*

PROOF. We wish to inscribe a square ($\boxdot ABCD$) in a given circle ($\circ ABCD$).

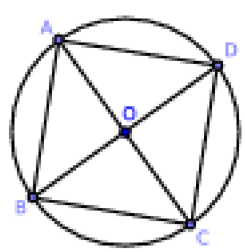

FIGURE 4.2.9. [4.6]

[1] https://en.wikipedia.org/wiki/Nine-point_circle

[2] https://en.wikipedia.org/wiki/Leonhard_Euler

Construct any two diameters AC, BD such that $AC \perp BD$. Join AB, BC, CD, DA. We claim that $\boxdot ABCD$ is the required square.

Let O be the center of $\circ ABCD$. Then the four angles at O are equal since they are right angles. Hence the arcs on which they stand are equal [3.26] and the four chords on which they stand are equal in length [3.29]. Therefore the figure $\boxdot ABCD$ is equilateral. Again, because AC is a diameter, the angle $\angle ABC$ is right [3.31]. Similarly, the remaining angles are right. It follows that $\boxdot ABCD$ is a square inscribed in $\circ ABCD$. □

PROPOSITION 4.7. *CIRCUMSCRIBE A SQUARE ABOUT A CIRCLE. It is possible to circumscribe a square about a given circle.*

PROOF. We wish to construct a circle ($\circ ABC$) about the square ($\boxdot ABCD$).

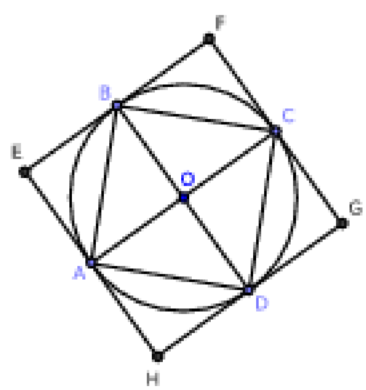

FIGURE 4.2.10. [4.7]

Through the center O construct any two diameters at right angles to each other, and at the points A, B, C, D construct the tangential segments HE, EF, FG, GH. We claim that $\boxdot EFGH$ is the required square.

Since AE touches the circle at A, the angle $\angle EAO$ is right [3.18] and therefore equal to $\angle BOC$, which is right by construction. Hence $AE \parallel OB$. Similarly, $EB \parallel AO$. Since $AO = OB$, the figure $AOBE$ is a lozenge. Since the angle $\angle AOB$ is right, $\boxdot AOBE$ is a square. Similarly, each of the figures $\boxdot BOFC$, $\boxdot ODGC$, $\boxdot AHDO$ is a square, and so $\boxdot EFGH$ is a square. □

COROLLARY. *1. The circumscribed square has double the area of the inscribed square.*

PROPOSITION 4.8. *INSCRIBE A CIRCLE IN A SQUARE. It is possible to inscribe a circle in a given square.*

PROOF. We wish to inscribe a circle ($\circ ABC$) in a given square ($\boxdot EFGH$).

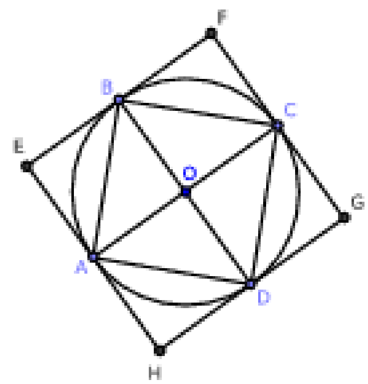

FIGURE 4.2.11. [4.8]

Bisect the two adjacent sides EH, EF at the points A, B, and through A, B construct the segments AC, BD which are respectively parallel to EF, EH. We claim that O, the point of intersection of these parallels, is the center of the required circle $\circ ABC$.

Because $\boxdot AOBE$ is a parallelogram, its opposite sides are equal; therefore $AO = EB$. But EB is half the side of the given square, and so AO is equal to half the side of the given square. This is similarly true for each of the segments OB, OC, OD. Therefore we have that $OA = OB = OC = OD$. Since they are perpendicular to the sides of the given square, the circle constructed with O as center and OA as radius is inscribed in the square. □

PROPOSITION 4.9. *CIRCUMSCRIBE A CIRCLE ABOUT A GIVEN SQUARE. It is possible to circumscribe a circle about a given square.*

PROOF. We wish to construct a circle ($\circ ABC$) about a given square ($\boxdot ABCD$).

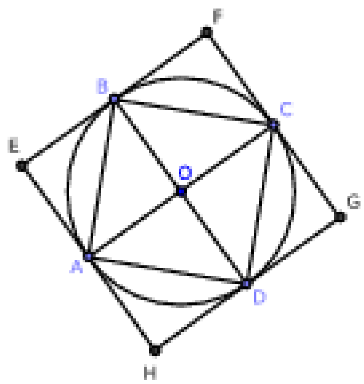

FIGURE 4.2.12. [4.9]

Construct perpendicular diagonals AC, BD intersecting at O. We claim that O is the center of the required circle $\circ ABC$.

Since $\triangle ABC$ is an isosceles triangle and the angle $\angle ABC$ is right, each of the other angles equals half a right angle; therefore $\angle BAO$ equals half a right angle. Similarly, $\angle ABO$ equals half a right angle; hence $\angle BAO = \angle ABO$. By [1.6], $AO = OB$. Similarly, $OB = OC$ and $OC = OD$. Hence the circle with O as center and OA as radius intersects through the points B, C, D and is evidently constructed about the square $\boxdot ABCD$. □

PROPOSITION 4.10. *CONSTRUCTION OF AN ISOSCELES TRIANGLE WITH BASE ANGLES DOUBLE THE VERTICAL ANGLE. It is possible to construct an isosceles triangle such that each base angle is double the vertical angle.*

PROOF. We wish to construct an isosceles triangle ($\triangle ABD$) where

$$\angle ADB = \angle ABD = 2 \cdot \angle DAB$$

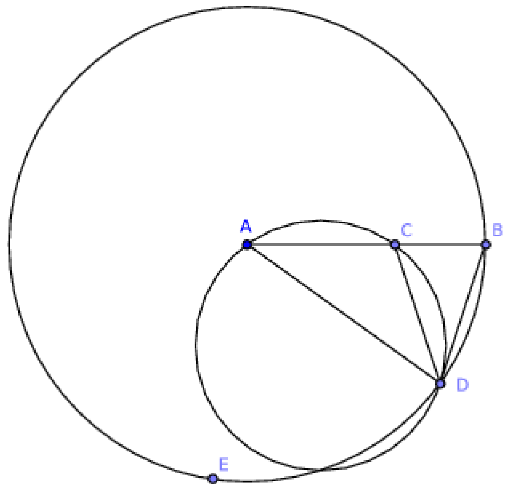

FIGURE 4.2.13. [4.10]

Take any segment AB and divide it at C such that the rectangle $AB.BC = AC^2$ [2.6]. With A as center and AB as radius, construct the circle $\circ BDE$ and on its circumference construct the chord $BD = AC$ [4.1]. Join AD. We claim that $\triangle ABD$ fulfills the required conditions.

Join CD. About the triangle $\triangle ACD$, construct the circle $\circ ACD$ [4.5]. Because by construction the rectangle $AB.BC = AC^2$ and $AC = BD$, each by construction, we have that the rectangle $AB.BC = BD^2$. By [3.32], BD touches the circle $\circ ACD$. Hence the angle $\angle BDC$ is equal to the angle at A in the alternate segment (or, $\angle BDC = \angle DAB$) [3.32].

To each angle, add $\angle CDA$, and we have that $\angle BDA = \angle CDA + \angle DAB$. But the exterior angle $\angle BCD$ of the triangle $\triangle ACD$ is equal to the sum of the angles $\angle CDA$ and $\angle CAD$ (or $\angle BCD = \angle CDA + \angle DAB$). Hence $\angle BDA = \angle BCD$. Since $AB = AD$, we have that $\angle BDA = \angle ABD$, from which it follows that

$$\angle BCD = \angle ABD = \angle CBD$$

By [1.6], $BD = CD$; but $BD = AC$ by construction. Therefore $AC = CD$, and by [1.5], $\angle CDA = \angle DAB$. But since $\angle BDA = \angle CDA + \angle DAB$, $\angle BDA = 2 \cdot \angle DAB$. It follows that each of the base angles of the triangle $\triangle ABD$ is double of the vertical angle. □

Exercises.

1. Prove that $\triangle ACD$ is an isosceles triangle whose vertical angle is equal to three times each of the base angles.

2. Prove that BD is the side of a regular decagon inscribed in the circle $\circ BDE$.

3. If DB, DE, EF are consecutive sides of a regular decagon inscribed in a circle, prove that $BF – BD =$ radius of a circle.

4. If E is the second point of intersection of the circle $\circ ACD$ with $\circ BDE$, then $DE = DB$. If AE, BE, CE, DE are joined, each of the triangles $\triangle ACE$, $\triangle ADE$ is congruent with $\triangle ABD$.

5. AC is the side of a regular pentagon inscribed in the circle $\circ ACD$, and EB the side of a regular pentagon inscribed in the circle $\circ BDE$.

6. Since $\triangle ACE$ is an isosceles triangle, $EB^2 – EA^2 = AB.BC = BD^2$; that is, the square of the side of a pentagon inscribed in a circle exceeds the square of the side of the decagon inscribed in the same circle by the square of the radius.

PROPOSITION 4.11. *INSCRIBE A REGULAR PENTAGON IN A GIVEN CIRCLE. It is possible to inscribe a regular pentagon in a given circle.*

PROOF. We wish to inscribe a regular pentagon in a given circle ($\circ ABC$).

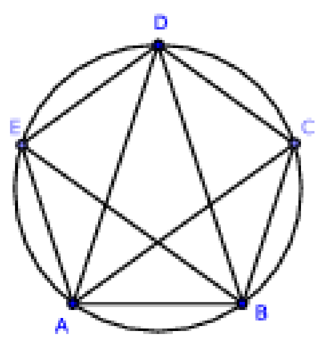

FIGURE 4.2.14. [4.11]

Construct an isosceles triangle having each base angle double the vertical angle [4.10], and inscribe a triangle $\triangle ABD$ equiangular to it in the given circle $\circ ABC$. Bisect the angles $\angle DAB$, $\angle ABD$ by the segments AC, BE. Join EA, ED, DC, CB. We claim that the figure $ABCDE$ is a regular pentagon.

Since each of the base angles $\angle BAD$, $\angle ABD$ is double of the angle $\angle ADB$ (or $\angle BAD = 2 \cdot \angle ADB = \angle ABD$) and the segments AC, BE bisect them, we have that

$$\angle BAC = \angle CAD = \angle ADB = \angle DBE = \angle EBA$$

Therefore the arcs on which they stand are equal, and the five chords AB, BC, CD, DE, EA are equal in length. Hence, the figure $ABCDE$ is equilateral.

Again, because the arcs AB, DE are equal, if we add the arc BCD to both, the arc $ABCD$ is equal to the arc $BCDE$, and therefore the angles $\angle AED$, $\angle BAE$ which stand on them are equal [3.27]. Similarly, it can be shown that all the angles are equal; therefore $ABCDE$ is equiangular. Hence, it is a regular pentagon. □

Exercises.

1. The figure formed by the five diagonals of a regular pentagon is another regular pentagon.

2. If the alternate sides of a regular pentagon are extended to meet, the five points of meeting form another regular pentagon.

3. Every two consecutive diagonals of a regular pentagon divide each other in the extreme and mean ratio.

4. Being given a side of a regular pentagon, construct it.

5. Divide a right angle into five equal parts.

PROPOSITION 4.12. *CIRCUMSCRIBE A REGULAR PENTAGON ABOUT A GIVEN CIRCLE. It is possible to circumscribe a regular pentagon about a given circle.*

PROOF. We wish to construct a regular pentagon about a given circle ($\circ ABC$).

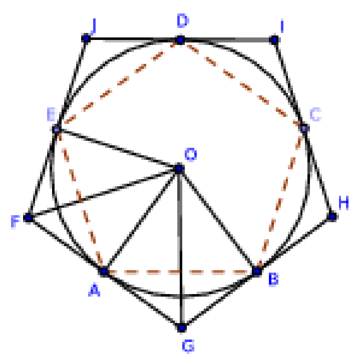

FIGURE 4.2.15. [4.12]

Let the five points A, B, C, D, E on the circumference $\circ ABC$ be the vertices of any inscribed regular pentagon; at these points, construct tangents FG, GH, HI, IJ, JF. We claim that $FGHIJ$ is a circumscribed regular pentagon.

Let O be the center of $\circ ABC$. Join OE, OA, OB. Because the angles $\angle OAF$, $\angle OEF$ of the quadrilateral $AOEF$ are right angles [3.18], the sum of the two remaining angles $\angle AOE + \angle AFE$ equals two right angles. Similarly, the sum $\angle AOB + \angle AGB$ equals two right angles; hence

$$\angle AOE + \angle AFE = \angle AOB + \angle AGB$$

But we have that $\angle AOE = \angle AOB$ because they stand on equal arcs AE, AB [3.27]. Hence $\angle AFE = \angle AGB$. Similarly, the remaining angles of the figure $FGHIJ$ are equal, and so $FGHIJ$ is equiangular.

Again, join OF, OG. Notice that the triangles $\triangle EOF$, $\triangle AOF$ share equal sides AF, FE [3.17, #1], have side FO in common, and have equal bases AO, EO. Hence $\triangle EOF \cong \triangle AOF$, and so $\angle AFO = \angle EFO$ [1.8]. Therefore $\angle AFO = \frac{1}{2}\angle AFE$. Similarly, $\angle AGO = \frac{1}{2}\angle AGB$. But $\angle AFE = \angle AGB$, and so $\angle AFO = \angle AGO$; also, $\angle FAO = \angle GAO$ since each are right angles, and the two triangles $\triangle FAO$, $\triangle GAO$ share side AO in common. By [1.26], $AF = AG$, and so $GF = 2 \cdot AF$; similarly, $JF = 2 \cdot EF$. And since $AF = EF$, $GF = JF$.

Similarly, the remaining sides are equal; therefore the figure $FGHIJ$ is equilateral and equiangular. Hence, it is a regular pentagon. □

Note: This proposition is a particular case of the following general theorem (which has a analogous proof): "If tangents are constructed to a circle at the angular points of an inscribed polygon of any number of sides, they will form a regular polygon of the same number of sides circumscribed to the circle."

PROPOSITION 4.13. *INSCRIBING A CIRCLE IN A REGULAR PENTAGON. It is possible to inscribe a circle in a regular pentagon.*

PROOF. We wish to inscribe a circle ($\circ JFG$) in a regular pentagon ($ABCDE$).

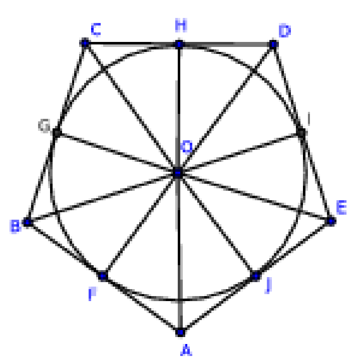

FIGURE 4.2.16. [4.13]

Bisect two adjacent angles $\angle JAF$, $\angle FBG$ by the segments AO, BO; we claim that the point of intersection of the bisectors, O, is the center of the required circle $\circ JFG$.

Join CO, and construct perpendiculars from O on the five sides of the pentagon. Notice that the triangles $\triangle ABO$, $\triangle CBO$ have the sides $AB = BC$ by hypothesis, side BO in common, and $\angle ABO = \angle CBO$ by construction. By [1.4], $\angle BAO = \angle BCO$; however, $\angle BAO = \frac{1}{2}\angle BAE$ by construction. Therefore

$$\angle BCO = \frac{1}{2}\angle BAE = \frac{1}{2}\angle BCD$$

and hence CO bisects the angle $\angle BCD$.

Similarly, it may be proved that DO bisects the angle $\angle HDI$ and EO bisects the angle $\angle IEJ$. Again, the triangles $\triangle BOF$, $\triangle BOG$ have the angle $\angle OFA = \angle OGB$ since each are right, $\angle OBF = OBG$ because OB bisects $\angle ABC$ by construction, and they share side OB in common. Hence, $OF = OG$ [1.26].

Similarly, all the perpendiculars from O on the sides of the pentagon are equal. Hence the circle whose center is O with radius OF touches all the sides of the pentagon and will therefore be inscribed in it; or, a circle may be inscribed in any regular polygon. □

PROPOSITION 4.14. *CIRCUMSCRIBE A CIRCLE ABOUT A REGULAR PENTAGON. It is possible to circumscribe a circle about a regular pentagon.*

PROOF. We wish to construct a circle ($\circ AED$) about a regular pentagon ($ABCDE$).

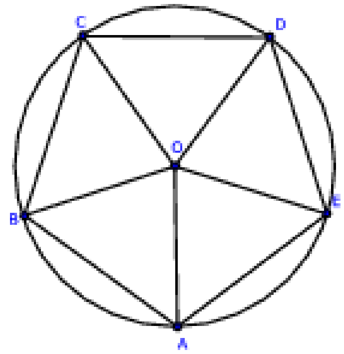

FIGURE 4.2.17. [4.14]

Bisect two adjacent angles $\angle BAE$, $\angle ABC$ by the segments AO, BO. We claim that O, the point of intersection of the bisectors, is the center of the required circle $\circ ABC$.

Join OC, OD, OE. Then the triangles $\triangle ABO$, $\triangle CBO$ have the side $AB = BC$ by hypothesis, BO common, and $\angle ABO = \angle CBO$ by construction. By [1.4], $\angle BAO = \angle BCO$. But $\angle BAE = \angle BCD$ by hypothesis. Since $\angle BAO = \frac{1}{2}\angle BAE$ by construction, $\angle BCO = \frac{1}{2}\angle BCD$. Hence CO bisects the angle $\angle BCD$.

Similarly, it may be proved that DO bisects $\angle CDE$ and EO bisects the angle $\angle DEA$. Again, because $\angle EAB = \angle ABC$, their halves are equal, and $\angle OAB = \angle OBA$. By [1.4], $OA = OB$. Similarly, the segments OC, OD, OE are equal to one another and to OA. Therefore the circle constructed with O as center and OA as radius ($\circ AED$) passes through the points B, C, D, E and is constructed about the pentagon. □

Note: [4.13] and [4.14] are particular cases of the following theorem: "A regular polygon of any number of sides has one circle inscribed in it and another constructed about it, and both circles are concentric.

PROPOSITION 4.15. *INSCRIBE A REGULAR HEXAGON IN A CIRCLE. It is possible to inscribe a regular hexagon in a circle.*

PROOF. We wish to to inscribe a regular hexagon ($ABCDEF$) in a given circle ($\circ ABC$).

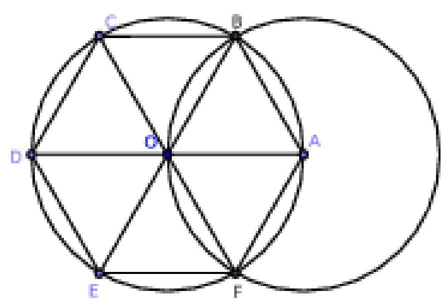

FIGURE 4.2.18. [4.15]

Take any point A on the circumference and join it to O, the center of the circle. Then with A as center and AO as radius, construct the circle $\circ OBF$, intersecting $\circ ABC$ at the points B, F. Join OB, OF and extend AO, BO, FO to meet $\circ ABC$ again at the points D, E, C. Join AB, BC, CD, DE, EF, FA; we claim that $ABCDEF$ is the required hexagon.

Each of the triangles $\triangle AOB$, $\triangle AOF$ is equilateral. Hence the angles $\angle AOB$, $\angle AOF$ are each one-third of two right angles; therefore $\angle EOF$ is one-third of two right angles. Again, the angles $\angle BOC$, $\angle COD$, $\angle DOE$ are respectively equal to the angles $\angle EOF$, $\angle FOA$, $\angle AOB$ [1.15]. Therefore the six angles at the center are equal, because each is one-third of two right angles, and so by [3.29],

$$AB = BC = CD = DE = EF = FA$$

Hence the hexagon is equilateral.

Since the arc $AF = ED$, to each arc add arc $ABCD$; it follows that the whole arc $FABCD = ABCDE$, and therefore the angles $\angle DEF$, $\angle EFA$ which stand on these arcs are equal [3.27]. Similarly, it may be shown that the other angles of the hexagon are equal. Hence $ABCDE$ is equiangular and is therefore a regular hexagon inscribed in the circle. □

COROLLARY. *1. The length of the side of a regular hexagon inscribed in a circle is equal to the circle's radius.*

COROLLARY. *2. If three alternate angles of a hexagon are joined, they form an inscribed equilateral triangle.*

Exercises.

1. The area of a regular hexagon inscribed in a circle is equal to twice the area of an equilateral triangle inscribed in the circle. Also, the square of the side of the triangle equals three times the square of the area of the side of the hexagon.

2. If the diameter of a circle is extended to C until the extended segment is equal to the radius, then the two tangents from C and their chord of contact form an equilateral triangle.

3. The area of a regular hexagon inscribed in a circle is half the area of an equilateral triangle and three-fourths of the area of a regular hexagon circumscribed to the circle.

PROPOSITION 4.16. *INSCRIBE A REGULAR FIFTEEN-SIDED POLYGON IN A GIVEN CIRCLE. It is possible to inscribe a regular, fifteen-sided polygon in a given circle.*

PROOF. We wish to inscribe a regular fifteen-sided polygon in a given circle ($\circ ABC$).

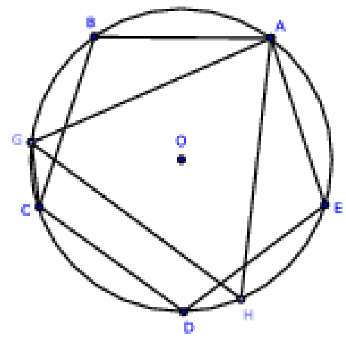

FIGURE 4.2.19. [4.16]

Inscribe a regular pentagon $ABCDE$ in the circle $\circ ABC$ [4.11] and also an equilateral triangle $\triangle AGH$ [4.2]. Join CG. We claim that CG is a side of the required polygon.

Since $ABCDE$ is a regular pentagon, the arc ABC is $2/5$ of the circumference. Since $\triangle AGH$ is an equilateral triangle, the arc ABG is $1/3$ of the circumference. Hence the arc GC, which is the difference between these two arcs, is equal to $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$ of the entire circumference. Therefore, if chords equal to GC are placed round the circle [4.1], we have a regular fifteen-sided polygon, or quindecagon, inscribed in it. □

Note: Until the year 1801, no regular polygon could be constructed by lines and circles only except those discussed in this Book IV of Euclid and those obtained from them by the continued bisection of the arcs of which their sides are the chords; but in that year, Gauss proved that if $2n+1$ is a prime number, then regular polygons of $2n+1$ sides are constructible by elementary geometry.

Examination question for chapter 4.

1. What is the subject-matter of chapter 4?
2. When is one polygon said to be inscribed in another?
3. When is one polygon said to be circumscribed about another?
4. When is a circle said to be inscribed in a polygon?
5. When is a circle said to be circumscribed about a polygon?
6. What is meant by reciprocal propositions? (Ans. In reciprocal propositions, to every line in one there corresponds a point in the other; and, conversely, to every point in one there corresponds a line in the other.)
7. Give instances of reciprocal propositions in chapter 4.

8. What is a regular polygon?

9. What figures can be inscribed in, and circumscribed about, a circle by means of chapter 4?

10. What regular polygons has Gauss proved to be constructible by the line and circle?

11. What is meant by escribed circles?

12. How many circles can be constructed to touch three lines forming a triangle?

13. What is the centroid of a triangle?

14. What is the orthocenter?

15. What is the circumcenter?

16. What is the polar circle?

17. What is the "nine-points circle"?

18. How does a nine-points circle get its name?

19. Name the nine points that a nine-points circle passes through.

20. What three regular figures can be used in filling up the space round a point? (Ans. Equilateral triangles, squares, and hexagons.)

21. If the sides of a triangle are 13, 14, 15 units in length, what are the values of the radii of its inscribed and escribed circles?

22. What is the radius of the circumscribed circle?

23. What is the radius of its nine-points circle?

24. What is the distance between the centers of its inscribed and circumscribed circles?

25. If r is the radius of a circle, what is the area:

(a) of its inscribed equilateral triangle?

(b) of its inscribed square?

(c) its inscribed pentagon?

(d) its inscribed hexagon?

(e) its inscribed octagon?

(f) its inscribed decagon?

26. With the same hypothesis, find the sides of the same regular figures.

Exercises on chapter 4.

1. If a circumscribed polygon is regular, the corresponding inscribed polygon is also regular, and conversely.

2. If a circumscribed triangle is isosceles, the corresponding inscribed triangle is isosceles, and conversely.

3. If the two isosceles triangles in #2 have equal vertical angles, they are both equilateral.

4. Divide an angle of an equilateral triangle into five equal parts.

5. Inscribe a circle in a sector of a given circle.

6. Suppose that segments DE, BC of $\triangle ABC$ are parallel: $DE \parallel BC$. Prove that the circles constructed about the triangles $\triangle ABC$, $\triangle ADE$ touch at A.

7. If the diagonals of a cyclic quadrilateral intersect at E, prove that the tangent at E to the circle about the triangle $\triangle ABE$ is parallel to CD.

8. Inscribe a regular octagon in a given square.

9. If a segment of given length slides between two given lines, find the locus of the intersection of perpendiculars from its endpoints to the given lines.

10. If the perpendicular to any side of a triangle at its midpoint meets the internal and external bisectors of the opposite angle at the points D and E, prove that D, E are points on the circumscribed circle.

11. Through a given point P, construct a chord of a circle so that the intercept EF stands opposite a given angle at point X.

12. In a given circle, inscribe a triangle having two sides passing through two given points and the third parallel to a given line.

13. Given four points, no three of which are collinear, construct a circle which is equidistant from them.

14. In a given circle, inscribe a triangle whose three sides pass through three given points.

15. Construct a triangle, being given:

(a) the radius of the inscribed circle, the vertical angle, and the perpendicular from the vertical angle on the base.

(b) the base, the sum or difference of the other sides, and the radius of the inscribed circle, or of one of the escribed circles.

(c) the centers of the escribed circles.

16. If F is the midpoint of the base of a triangle, DE the diameter of the circumscribed circle which passes through F, and L the point where a parallel to the base through the vertex meets DE, prove that $DL.FE$ is equal to the square of half the sum and $DF.LE$ is equal to the square of half the difference of the two remaining sides.

17. If from any point within a regular polygon of n sides perpendiculars fall on the sides, their sum is equal to n times the radius of the inscribed circle.

18. The sum of the perpendiculars falling from the angular points of a regular polygon of n sides on any line is equal to n times the perpendicular from the center of the polygon on the same line.

19. If R denotes the radius of the circle circumscribed about a triangle $\triangle ABC$, r, r', r'', r''' are the radii of its inscribed and escribed circles; δ, δ', δ'' are the perpendiculars from its circumcenter on the sides; μ, μ', μ'' are the

segments of these perpendiculars between the sides and circumference of the circumscribed circle, then we have the equalities:

$$\begin{aligned} r' + r'' + r''' &= 4R + r \quad (1) \\ \mu + \mu' + \mu'' &= 2R - r \quad (2) \\ \delta + \delta' + \delta'' &= R + r \quad (3) \end{aligned}$$

The relation (3) supposes that the circumcenter is inside the triangle.

20. Take a point D from the side BC of a triangle $\triangle ABC$ and suppose we construct a transversal EDF through it; suppose we also construct circles about the triangles $\triangle DBF$, $\triangle ECD$. The locus of their second point of intersection is a circle.

21. In every quadrilateral circumscribed about a circle, the midpoints of its diagonals and the center of the circle are collinear.

22. Find on a given line a point P, the sum or difference of whose distances from two given points may be given.

23. Find a point such that, if perpendiculars fall from it on four given lines, their feet are collinear.

24. The line joining the orthocenter of a triangle to any point P in the circumference of its circumscribed circle is bisected by the line of co-linearity of perpendiculars from P on the sides of the triangle.

25. The orthocenters of the four triangles formed by any four lines are collinear.

26. If a semicircle and its diameter are touched by any circle either internally or externally, then twice the rectangle contained by the radius of the semicircle and the radius of the tangential circle is equal to the rectangle contained by the segments of any secant to the semicircle through the point of intersection of the diameter and touching circle.

27. If ρ, ρ' are radii of two circles touching each other at the center of the inscribed circle of a triangle where each touches the circumscribed circle, prove that

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2}{r}$$

and state and prove corresponding theorems for the escribed circles.

28. If from any point in the circumference of the circle, circumscribed about a regular polygon of n sides, segments are constructed to its angular points, the sum of their squares is equal to $2n$ times the square of the radius.

29. In the same case as the above, if the lines are constructed from any point in the circumference of the inscribed circle, prove that the sum of their

squares is equal to n times the sum of the squares of the radii of the inscribed and the circumscribed circles.

30. State the corresponding theorem for the sum of the squares of the lines constructed from any point in the circumference of any concentric circle.

31. If from any point in the circumference of any concentric circle perpendiculars are let fall on all the sides of any regular polygon, the sum of their squares is constant.

32. See #31. For the inscribed circle, the constant is equal to $3n/2$ times the square of the radius.

33. See #31. For the circumscribed circle, the constant is equal to n times the square of the radius of the inscribed circle, together with $\frac{1}{2}n$ times the square of the radius of the circumscribed circle.

34. If the circumference of a circle whose radius is R is divided into seventeen equal parts and AO is the diameter constructed from one of the points of division (A), and if ρ_1, ρ_2, ..., ρ_8 denote the chords from O to the points of division, A_1, A_2, ..., A_8 on one side of AO, then

$$\rho_1\rho_2\rho_4\rho_8 = \rho_3\rho_5\rho_6\rho_7 = R^4$$

Let the supplemental chords corresponding to ρ_1, ρ^2, etc., be denoted by r_1, r_2, etc. Then by [3.35, #2], we have that

$$\begin{aligned} \rho_1 r_1 &= Rr_2 \\ \rho_2 r_2 &= Rr_4 \\ \rho_4 r_4 &= Rr_8 \\ \rho_8 r_8 &= Rr_1 \end{aligned}$$

Hence, $\rho_1\rho_2\rho_4\rho_8 = R^4$.

And it may be similarly proved that

$$\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6\rho_7\rho_8 = R^8$$

Therefore, $\rho_3\rho_5\rho_6\rho_7 = R^4$.

35. If from the midpoint of the segment joining any two of four concyclic points a perpendicular falls on the line joining the remaining two, the six perpendiculars thus obtained are concurrent.

36. The greater the number of sides of a regular polygon circumscribed about a given circle, the less will be its perimeter.

37. The area of any regular polygon of more than four sides circumscribed about a circle is less than the square of the diameter.

38. Four concyclic points taken three by three determine four triangles, the centers of whose nine-points circles are concyclic.

39. If two sides of a triangle are given in position, and if their included angle is equal to an angle of an equilateral triangle, the locus of the center of its nine-points circle is a straight line.

40. If in the hypothesis and notation of #34, α, β denote any two suffixes whose sum is less than 8 where $\alpha > \beta$, then

$$\rho_\alpha \rho_\beta = R(\rho_{\alpha-\beta} + \rho_{\alpha+\beta})$$

For example, $\rho_1\rho_4 = R(\rho_3 + \rho_5)$ [3.35, #7].

In the same case, if the suffixes be greater than 8, then

$$\rho_\alpha \cdot \rho_\beta = R(\rho_{\alpha-\beta} - \rho_{17-\alpha-\beta})$$

For instance, $\rho_8 \cdot \rho_2 = R(\rho_6 - \rho_7)$ [3.35, #6].

41. Two lines are given in position. Construct a transversal through a given point, forming with the given lines a triangle of given perimeter.

42. Given the vertical angle and perimeter of a triangle, construct it with any of the following data:

(a) The bisector of the vertical angle;

(b) the perpendicular from the vertical angle on the base;

(c) the radius of the inscribed circle.

43. In a given circle, inscribe a triangle so that two sides may pass through two given points and that the third side may be a maximum or a minimum.

44. If s is the semi-perimeter of a triangle, and if r', r'', r''' are the radii of its escribed circles, then

$$r' \cdot r'' + r'' \cdot r''' + r''' \cdot r' = s^2$$

45. The feet of the perpendiculars from the endpoints of the base on either bisector of the vertical angle, the midpoint of the base, and the foot of the perpendicular from the vertical angle on the base are concyclic.

46. Given the base of a triangle and the vertical angle, find the locus of the center of the circle passing through the centers of the escribed circles.

47. The perpendiculars from the centers of the escribed circles of a triangle on the corresponding sides are concurrent.

48. If AB is the diameter of a circle, PQ is any chord cutting AB at O, and if the segments AP, AQ intersect the perpendicular to AB at O (at D and E respectively), then the points A, B, D, E are concyclic.

49. If the sides of a triangle are in arithmetical progression, and if R, r are the radii of the circumscribed and inscribed circles, then $6Rr$ is equal to the rectangle contained by the greatest and least sides.

50. Inscribe in a given circle a triangle having its three sides parallel to three given lines.

51. If the sides AB, BC, etc., of a regular pentagon is bisected at the points A', B', C', D', E', and if the two pairs of alternate sides BC, AE and AB, DE meet at the points A'', E'', respectively, prove that

$$\triangle A''AE'' - \triangle A'AE' = \text{pentagon } A'B'C'D'E'$$

52. In a circle, prove that an equilateral inscribed polygon is regular; also prove that if the number of its sides are odd, then it is an equilateral circumscribed polygon.

53. Prove that an equiangular circumscribed polygon is regular; also prove that if the number of its sides are odd, then it is an equilateral inscribed polygon.

54. The sum of the perpendiculars constructed to the sides of an equiangular polygon from any point inside the figure is constant.

55. Express the sides of a triangle in terms of the radii of its escribed circles.

CHAPTER 5

Theory of Proportions

Chapter 5, like Chapter 2, proves a number of propositions which demonstrate elementary algebraic statements that are more familiar to us in the form of equations. Algebra as we know it had not been developed when Euclid wrote "The Elements". As such, the results are more of historical importance than practical use (except when they are used in subsequent proofs). As such, Book V appears here in truncated form.

5.1. Definitions

1. A lesser magnitude is said to be a *part* or *submultiple* of a greater magnitude when the lesser magnitude is contained an exact number of times in the greater magnitude.

2. A greater magnitude is said to be a *multiple* of a lesser magnitude when the greater magnitude contains the lesser magnitude an exact number of times.

3. A *ratio* is the mutual relation of two magnitudes of the same kind with respect to quantity.

4. Magnitudes are said to have a *ratio* to one another when the lesser magnitude can be multiplied so as to exceed the greater.

These definitions require explanation, especially [Def. 5.3], which has the fault of conveying no precise meaning—being, in fact, unintelligible. We reconsider the above using algebraic terminology:

(a) If an integer is divided into any number of equal parts, then one part or the sum of any number of these parts is called a *fraction*.

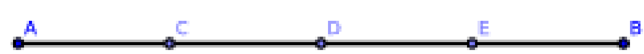

FIGURE 5.1.1. Def. 5.1

If the segment AB represents the integer, and if it is divided into four equal parts at the points C, D, E, then $AC = 1/4$, $AD = 2/4$, and $AE = 3/4$ of the whole segment, AB. Thus, a fraction is denoted by two numbers denoted above and below by a horizontal segment; the lower, *the denominator*, denotes the number of equal parts into which the integer is divided, and the upper, *the numerator*, denotes the number of these equal parts which are taken.

It follows that if the numerator is less than the denominator, the fraction is less than 1. If the numerator is equal to the denominator, the fraction equals to 1. And if the numerator is greater than the denominator, the fraction is greater than 1. It is evident that a fraction is an abstract quantity; that is, that its value is independent of the nature of the integer which is being divided.

(b) If we divide each of the equal parts AC, CD, DE, EB into two equal parts, the whole, AB, will be divided into eight equal parts, and we see that $AC = 2/8$, $AD = 4/8$, $AE = 6/8$, and $AB = 8/8$.

As we saw above, $AE = 3/4$ of the integer AB, and we have just demonstrated that $AE = 6/8$. Hence, $3/4 = 6/8$, which we can also obtain by multiplying the fraction $3/4$ by $2/2$, or

$$\frac{3}{4} = \frac{3}{4} \cdot 1 = \frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}$$

Hence we infer generally that multiplying each of the terms of any fraction by 2 does not alter its value. Similarly, it may be shown that multiplying each of the terms of a fraction by any nonzero integer does not alter its value. It follows conversely that dividing each of the terms of a fraction by a nonzero integer does not alter the value. Hence we have the following important and fundamental theorem: “The terms of a fraction can be either both multiplied or both divided by any nonzero integer and in either case the value of the new fraction is equal to the value of the original fraction.”

(c) If we take any number, such as 3, and multiply it by any nonzero integer, the product is called a multiple of 3. Thus $6, 9, 12, 15, ...$ are multiples of 3; but $10, 13, 17, ...$ are not, because the multiplication of 3 by any nonzero integer will not produce them. Conversely, 3 is a submultiple or part of $6, 9, 12, 15, ...$ because it is contained in each of these without a remainder; but not of $10, 13, 17, ...$ because in each case it leaves a remainder.

(d) If we consider two magnitudes of the same kind, such as two segments AB, CD in Fig. 5.1.2, and if we suppose that $AB = \frac{3}{4}CD$, it is evident that if AB is divided into three equal parts and CD is divided into four equal parts that each of the parts into which AB is divided is equal in length to each of the parts into which CD is divided.

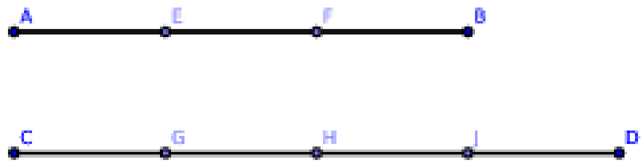

FIGURE 5.1.2.

As there are three parts in AB and four in CD, we express this relation by saying that AB has to CD the ratio of 3 to 4, and we denote it as $3 : 4$. Hence the ratio $3 : 4$ expresses the same idea as the fraction $3/4$. In fact, both are different ways of denoting the same thing: when written $3 : 4$, it is called a ratio, and when written as $3/4$, it is fraction. Similarly, it can be shown that every ratio whose terms are commensurable[1] can be converted into a fraction; and, conversely, every fraction can be turned into a ratio.

From this explanation, we see that the ratio of any two commensurable magnitudes is the same as the ratio of the numerical quantities which denote these magnitudes. Thus, the ratio of two commensurable lines is the ratio of the numbers which express their lengths measured with the same unit. And this may be extended to the case where the lines are incommensurable. If a is the side of a square and b is its diagonal, the ratio of $a : b$ is

$$\frac{a}{b} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

When two quantities are incommensurable, such as the diagonal and the side of a square, although their ratio is not equal to that of any two commensurable numbers, a series of pairs of fractions can be found whose difference is continually diminishing and which ultimately becomes infinitely small. We say that these fractions converge to an irrational number[2] such as $1/\sqrt{2}$. In this example, we have that:

$$\{\frac{1}{2}, \frac{14}{20}, \frac{141}{200}, \frac{1414}{2000}, \frac{14142}{20000}, ...\} \text{ converges to } \frac{1}{\sqrt{2}}$$

In decimal form, this gives us:

$$\{0, 0.7, 0.707, 0.7071, 0.707108, ...\} \text{ converges to } \frac{1}{\sqrt{2}}$$

[1] http://en.wikipedia.org/wiki/Commensurability_(mathematics)
[2] http://en.wikipedia.org/wiki/Irrational_number

By this method and in either form, we can approximate as nearly as we wish to the exact value of the ratio. It is evident we may continue either sequence as far as we please. For example, if we consider the first member of the fractional sequence as $\frac{m}{n}$, then we may write the second member as $\frac{10m+k}{10n}$ where k can always be determined.

Furthermore, in the case of two incommensurable quantities, two fractions $\frac{m}{n}$ and $\frac{m+1}{n}$ can always be found where n can be made as large as we wish where one fraction is greater than the irrational fraction and the other fraction is less.

To see this, let a and b be the incommensurable quantities. Then, if $n \neq m$, we cannot have that $na = mb$. Since any multiple of a must lie between two consecutive multiples of b, such as mb and $(m+1)b$, we have that $\frac{na}{mb} > 1$ and $\frac{na}{(m+1)b} < 1$. Hence, we obtain

$$\begin{array}{ccccc} \frac{na}{(m+1)b} & < & 1 & < & \frac{na}{ma} \\ \\ \frac{n}{m+1} & < & \frac{b}{a} & < & \frac{n}{m} \\ \\ \frac{m}{n} & < & \frac{a}{b} & < & \frac{m+1}{n} \end{array}$$

Since the difference between $\frac{(m+1)}{n}$ and $\frac{m}{n}$ is $\frac{1}{n}$ which grows small as n grows large, we have that the difference between the ratio of two incommensurable quantities and that of two commensurable numbers m and n can be made as small as we please. Ultimately, the ratio of incommensurable quantities may be regarded as the limit of the ratio of commensurable quantities.

(e) The two terms of a ratio are called the *antecedent* and the *consequent*. These correspond to the numerator and the denominator of a fraction.

(f) The reciprocal of a ratio is the ratio obtained by interchanging the antecedent and the consequent. Thus, $4 : 3$ is the reciprocal of the ratio $3 : 4$. Hence, we have the following theorem: "The product of a ratio and its reciprocal is 1."

(g) If we multiply any two numbers, as 5 and 7, by any number such as 4, the products 20, 28 are called *equimultiples* of 5 and 7. Similarly, 10 and 15 are *equimultiples* of 2 and 3, and 18 and 30 of 3 and 5, etc.

5. Ratios and fractions preserve order; that is, multiplying by positive numbers preserve equalities and inequalities.

6. Magnitudes which have the same ratio are called proportionals. When four magnitudes are proportionals, it is usually expressed by saying, “The first is to the second as the third is to the fourth.” As an example:

$$\frac{a}{b} = \frac{c}{d} \text{ because } \frac{a}{b} = \frac{e}{e} \cdot \frac{c}{d}$$

As a ratio, the above equality may be written as:

$$a : b :: c : d$$

7. When we have multiples of four magnitudes (taken as in [Def. 5.5]) such that the multiple of the first is greater than that of the second but the multiple of the third is not greater than the fourth, the first has to the second a greater ratio than the third has to the fourth. Or:

$$\frac{a}{b} > \frac{c}{d} \text{ because } a > c \text{ and } b \leq d$$

8. Magnitudes which have the same ratio are called *proportional*.

9. Proportions consist of at minimum three terms.

This definition has the same fault as some of the others: it is not a definition, but an inference. It occurs when the means in a proportion are equal, so that, in fact, there are four terms. As an illustration, let us take the numbers 4, 6, 9. Here the ratio of $4 : 6$ is $2/3$, and the ratio of $6 : 9$ is $2/3$, so that 4, 6, 9 are continued proportionals; but, in reality, there are four terms: the full proportion is $4 : 6 :: 6 : 9$.

10. When three magnitudes are continual proportionals, the first is said to have to the third the *duplicate* ratio of that which it has to the second.

11. When four magnitudes are continual proportionals, the first is said to have to the fourth the *triplicate* ratio of that which it has to the second.

12. When there is any number of magnitudes of the same kind greater than two, the first is said to have to the last the ratio compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, etc.

13. A ratio whose antecedent is greater than its consequent is called a ratio of greater inequality. A ratio whose antecedent is less than its consequent is called a ratio of lesser inequality.

14. *Harmonic division* of a segment AB means identifying two points C and D such that AB is divided internally and externally in the same ratio $\frac{CA}{CB} = \frac{DA}{DB}$.

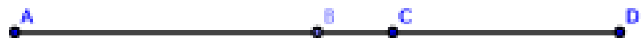

FIGURE 5.1.3. Here, the ratio is 2. Specifically, the distance AC is one unit, the distance CB is half a unit, the distance AD is three units, and the distance BD is 1.5 units.

Harmonic division of a line segment is reciprocal: if points C and D divide the segment AB harmonically, the points A and B also divide the line segment CD harmonically. In that case, the ratio is given by $\frac{BC}{BD} = \frac{AC}{AD}$ which equals one-third in the example above. (Note that the two ratios are not equal.)[3]

5.2. Propositions from Book V

PROPOSITION 5.1. *If any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.*

COROLLARY. *1. [5.1] is equivalent to* $kx + ky = k(x + y)$.

PROPOSITION 5.2. *If a first magnitude is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.*

COROLLARY. *1. [5.2] is equivalent to the following: if* $kv = x$, $kw = r$, $mv = y$, *and* $mw = u$, *then* $x + y = (k + m)v$ *and* $r + u = (k + m)w$.

PROPOSITION 5.3. *If a first magnitude is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the magnitudes taken also are equimultiples respectively, the one of the second and the other of the fourth.*

[3] http://en.wikipedia.org/wiki/Harmonic_division

COROLLARY. *1. [5.3] is equivalent to: let $A = kB$ and $C = kD$. Then if $EF = mA$ and $GM = mC$, then $EF = mkB$ and $GH = mkD$.*

PROPOSITION 5.4. *If a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.*

COROLLARY. *1. [5.4] is equivalent to: if $\frac{A}{B} = k = \frac{C}{D}$, then $A = kB$ and $C = kD$.*

PROPOSITION 5.5. *If a magnitude is the same multiple of a magnitude that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.*

COROLLARY. *1. [5.5] is equivalent to: if $x + y = k(m + n)$ and $x = km$, then $y = kn$.*

PROPOSITION 5.6. *If two magnitudes are equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.*

COROLLARY. *1. [5.6] is equivalent to: if $x + y = km$, $u + v = kn$, $x = lm$, $y = ln$, and all variables are positive, then $y = (k-l)m$ and $v = (k-l)n$ whenever $k > l$.*

PROPOSITION 5.7. *Equal magnitudes have to the same the same ratio; and the same has to equal magnitudes the same ratio.*

COROLLARY. *1. If any magnitudes are proportional, then they are also proportional inversely.*

COROLLARY. *2. [5.7] is equivalent to: if $a = b$, then $a = b = kc$, and $c = qa = qb$ where $q = \frac{1}{k}$.*

PROPOSITION 5.8. *Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.*

COROLLARY. *1. [5.8] is equivalent to: if $AB > C$ and $D > 0$, then $AB = C + k$ where $k > 0$, and $\frac{AB}{D} = \frac{C+k}{D} = \frac{C}{D} + \frac{k}{D} > \frac{C}{D}$. It follows that $\frac{D}{C} > \frac{D}{AB}$, since all quantities are positive.*

PROPOSITION 5.9. *Magnitudes which have the same ratio to the same equal one another; and magnitudes to which the same has the same ratio are equal.*

COROLLARY. *1. [5.9] is equivalent to: if $A = kC$ and $B = kC$, then $A = B$.*

PROPOSITION 5.10. *Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.*

COROLLARY. *1. [5.10] is equivalent to: if $\frac{A}{C} > \frac{B}{C}$ and $C > 0$, then $A > B$.*

PROPOSITION 5.11. *Ratios which are the same with the same ratio are also the same with one another.*

COROLLARY. *1. [5.11] is equivalent to: if $\frac{A}{B} = \frac{C}{D}$ and $\frac{C}{D} = \frac{E}{F}$, then $\frac{A}{B} = \frac{E}{F}$. This is the transitive property for fractions.*

PROPOSITION 5.12. *If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.*

COROLLARY. *1. [5.12] is equivalent to: if $A = kB$, $C = kD$, $E = kF$, then $A + C + E = kB + kD + kF = k(B + D + F)$.*

PROPOSITION 5.13. *If a first magnitude has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.*

COROLLARY. *1. [5.13] is equivalent to: if $A = kB$, $C = kD$, $C = lD$, $E = jF$ and $l > j$, then $k = l$ and so $k > j$.*

PROPOSITION 5.14. *If a first magnitude has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.*

COROLLARY. *1. [5.14] is equivalent to: if $A = kB$, $C = kD$, $A > C$, and $k > 0$, then $kB = A > C = kD$ and so $B > D$. If $A < C$ and $B < D$, the result follows mutatis mutandis.*

PROPOSITION 5.15. *Parts have the same ratio as their equimultiples.*

COROLLARY. *1. [5.15] is equivalent to: if $AB = kC$, $DE = kF$, and $C = mF$, then $AB = kmF = mDE$.*

PROPOSITION 5.16. *If four magnitudes are proportional, then they are also proportional alternately.*

COROLLARY. *1. [5.16] is equivalent to: if $\frac{A}{B} = \frac{C}{D}$, then $\frac{A}{C} = \frac{B}{D}$.*

PROPOSITION 5.17. *If magnitudes are proportional taken jointly, then they are also proportional taken separately.*

COROLLARY. *1. [5.17] is equivalent to: if $x + y = ky$, $u + v = kv$, and $x = ly$, then $ly + y = ky$, or $l + 1 = k$. Thus $u + v = (l + 1)v$, or $u = lv$.*

PROPOSITION 5.18. *If magnitudes are proportional taken separately, then they are also proportional taken jointly.*

COROLLARY. *1. [5.18] is equivalent to: if* $x = ky$*,* $u = kv$*, and* $x + y = ly$*, then* $(k+1)y = ly$ *and so* $k + 1 = l$ *and* $u + v = kv = (k+1)v = lv$*.*

PROPOSITION 5.19. *If a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.*

COROLLARY. *1. If magnitudes are proportional taken jointly, then they are also proportional in conversion.*

COROLLARY. *2. [5.19] is equivalent to: if* $x + y = k(u + v)$ *and* $x = ku$*, then* $y = kv$*.*

PROPOSITION 5.20. *If there are three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.*

COROLLARY. *1. [5.20] is equivalent to: let* $A = kB$*,* $B = lC$*,* $D = kE$*,* $E = lF$*, and* $A > C$*. We wish to show that* $D > F$*.*

Suppose that $A = c + m$*,* $m > 0$*. Then* $A = klC$*,* $D = klF$*, and so* $\frac{A}{C} = \frac{D}{F}$*.*

Now $\frac{A}{C} > 1$ *since* $A > C$*. If* $D = F$*,* $\frac{A}{C} = 1$*; and if* $D < F$*,* $\frac{A}{C} < 1$*. Hence,* $D > F$*.*

The remaining cases follow mutandis mutatis.

PROPOSITION 5.21. *If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if the first magnitude is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.*

COROLLARY. *1. The result of [5.21] is the same as the result [5.20].*

PROPOSITION 5.22. *If there are any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio.*

COROLLARY. *1. [5.22] is equivalent to: if* $A = kB$, $B = lC$, $D = kE$, *and* $E = lF$, *then* $A = klC$ *and* $D = klF$.

PROPOSITION 5.23. *If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio.*

COROLLARY. *1. The result of [5.23] is the same as the result of [5.22].*

PROPOSITION 5.24. *If a first magnitude has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.*

COROLLARY. *1. [5.24] is equivalent to: if* $x = km$, $u = kn$, $y = lm$, *and* $v = ln$, *then* $x + y = km + lm = (k + l)m$ *and* $u + v = kn + ln = (k + l)n$.

PROPOSITION 5.25. *If four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.*

COROLLARY. *1. [5.25] is equivalent to: let* $x + y = k(u + v)$, $k > 1$, *and* $x = ku$. *Since* $x + y = ku + kv$, $y = kv$; *and since* $k > 1$, $y > v$.

Examination questions for chapter 5.

1. What is the subject-matter of this chapter?
2. When is one magnitude said to be a multiple of another?
3. What is a submultiple or measure?
4. What are equimultiples?
5. What is the ratio of two commensurable magnitudes?

. What is meant by the ratio of incommensurable magnitudes?

7. Give an illustration of the ratio of incommensurables.
8. What are the terms of a ratio called?
9. What is duplicate ratio?
10. Define triplicate ratio.
11. What is proportion? (Ans. Equality of ratios.)
12. How many ratios in a proportion?

13. When is a segment divided harmonically?

14. What are reciprocal ratios?

Chapter 5 exercises.

1. A ratio of greater inequality is increased by diminishing its terms by the same quantity, and diminished by increasing its terms by the same quantity.

2. A ratio of lesser inequality is diminished by diminishing its terms by the same quantity, and increased by increasing its terms by the same quantity.

3. If four magnitudes are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

4. If two sets of four magnitudes are proportionals, and if we multiply corresponding terms together, the products are proportionals.

5. If two sets of four magnitudes are proportionals, and if we divide corresponding terms, the quotients are proportionals.

6. If four magnitudes are proportionals, their squares, cubes, etc., are proportionals.

7. If two proportions have three terms of one respectively equal to three corresponding terms of the other, the remaining term of the first is equal to the remaining term of the second.

8. If three magnitudes are continual proportionals, the first is to the third as the square of the difference between the first and second is to the square of the difference between the second and third.

9. If a line AB, cut harmonically at C and D, is bisected at O, prove that OC, OB, OD are continual proportionals.

10. In the same case, if O' is the midpoint of CD, prove that $OO'^2 = OB^2 + OD^2$.

11. Continuing from #10, show that $AB(AC + AD) = 2AC.AD$, or $\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$

12. Continuing from #10, show that $CD(AD + BD) = 2AD.BD$, or $\frac{1}{BD} + \frac{1}{AD} = \frac{2}{AC}$

13. Continuing from #10, show that $AB.CD = 2AD.CB$.

CHAPTER 6

Applications of Proportions

When comparing the proportions of areas of triangles, we will use the following abbreviation: if we wish to state that the area of $\triangle ABC$ divided by the area of $\triangle DEF$ is equal to the area of $\triangle GHI$ divided by the area of $\triangle JKL$, we will write

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\triangle GHI}{\triangle JKL}$$

or

$$\triangle ABC : \triangle DEF \quad :: \quad \triangle GHI : \triangle JKL$$

This is comparable for how we also write $\triangle GHI = \triangle JKL$ to denote that the area of $\triangle GHI$ is equal to the area of $\triangle JKL$.

6.1. Definitions

1. *Similar polygons* are those whose angles are respectively equal and whose sides about the equal angles are proportional. Similar figures agree in shape; if they also agree in size, then they are congruent. If polygons ABC and DEF are similar, we will denote this as $ABC \sim DEF$.

(a) When the shape of a figure is given, it is said to be given in species. Thus a triangle whose angles are given is given in species. Hence, similar figures are of the same species.

(b) When the size of a figure is given, it is said to be given in magnitude, such as a square whose side is of given length.

(c) When the place which a figure occupies is known, it is said to be given in position.

2. A segment is said to be cut at a point *in extreme and mean ratio* when the whole segment is to the greater segment as the greater segment is to the lesser segment.

3. If three quantities of the same kind are in continued proportion, the middle term is called a mean proportional between the other two. Magnitudes in continued proportion are also said to be in geometrical progression.

4. If four quantities of the same kind are in continued proportion, the two middle terms are called two mean proportionals between the other two.

5. The altitude of any figure is the length of the perpendicular from its highest point to its base.

6. Two corresponding angles of two figures have the sides about them *reciprocally proportional* when a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first.

This is equivalent to saying that a side of the first is to a side of the second in the reciprocal ratio of the remaining side of the first to the remaining side of the second.

7. *Similar figures* are said to be similarly constructed upon given segments when these lines are homologous[1] sides of the figures.

8. *Homologous points* in the planes of two similar figures are such that segments constructed from them to the angular points of the two figures are proportional to the homologous sides of the two figures. See Fig. 6.1.1.

9. The point O in Fig. 6.1.1 is called the center of similitude of the figures. It is also called their double point.

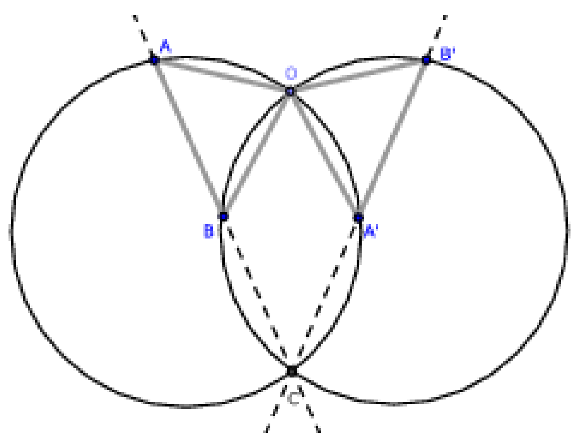

FIGURE 6.1.1. [Def 6.9] See also [6.20, #2]

6.2. Propositions from Book VI

PROPOSITION 6.1. *PROPORTIONAL TRIANGLES AND PARALLELOGRAMS. The areas of triangles and parallelograms which have the same altitude are proportional to their bases.*

[1]Def: “Having the same or a similar relation; corresponding, as in relative position or structure.” Dictionary.com, "homologous," in Dictionary.com Unabridged. Source location: Random House, Inc. http://dictionary.reference.com/browse/homologous. Available: http://dictionary.reference.com. Accessed: May 09, 2013.

PROOF. The areas of triangles ($\triangle ABC$, $\triangle ACD$) and of parallelograms ($\boxdot EBCA$, $\boxdot ACDF$) which share the same altitude are proportional to their bases (BC, CD).

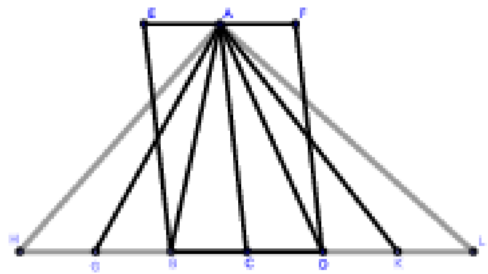

FIGURE 6.2.1. [6.1]

Construct $\triangle ACB$, $\boxdot EBCA$ on the base BC, and construct $\triangle ACD$, $\boxdot ACDF$ on the base CD such that each quadrilateral has the same altitude.

Extend BD in both directions to the points H and L. Construct any number of segments BG and GH which are equal in length to the base CB and any number of segments DK and KL which are equal in length to the base CD. Join AG, AH, AK, and AL [1.3]. Since $CB = BG = GH$, we obtain that $\triangle ACB = \triangle ABG = \triangle AGH$ [1.38]. Therefore, if $CH = k \cdot CB$ (where $k > 1$), we also have that $\triangle ACH = k \cdot \triangle ACB$.

Similarly, if $CL = m \cdot CD$ (where $m > 1$), we also have that $\triangle ACL = m \cdot \triangle ACD$. Finally, we also have that $CH = n \cdot CL$ (where $n > 0$) which implies that $\triangle ACH = n \cdot \triangle ACL$. Hence, we obtain

$$\begin{aligned} CH : CL &:: \triangle ACH : \triangle ACL && \Rightarrow \\ CH : m \cdot CD &:: \triangle ACH : m \cdot \triangle ACD && \Rightarrow \\ k \cdot CB : m \cdot CD &:: k \cdot \triangle ACB : m \cdot \triangle ACD && \Rightarrow \\ CB : CD &:: \triangle ACB : \triangle ACD \end{aligned}$$

Next, since $\boxdot EBCA = 2 \cdot \triangle ACB$ and $\boxdot ACDF = 2 \cdot \triangle ACD$ [1.41], we have that

$$\begin{aligned} 2 \cdot CB : 2 \cdot CD &:: \boxdot EBCA : \boxdot ACDF && \Rightarrow \\ CB : CD &:: \boxdot EBCA : \boxdot ACDF \end{aligned}$$

□

PROPOSITION 6.2. *PROPORTIONALITY OF SIDES OF TRIANGLES. If a segment is parallel to a side of a triangle, it divides the remaining sides proportionally (when measured from the opposite angle). Conversely, if two sides* *of a triangle, measured from an angle, are cut proportionally, the line joining the points of section is parallel to the third side.*

PROOF. If a segment (DE) is parallel to a side (BC) of a triangle ($\triangle ABC$), we claim that it divides the remaining sides proportionally (measured from the opposite angle, $\angle DAE$). Conversely, if two sides of a triangle, measured from an angle are cut proportionally, we claim that the segment joining the points of the section is parallel to the third side.

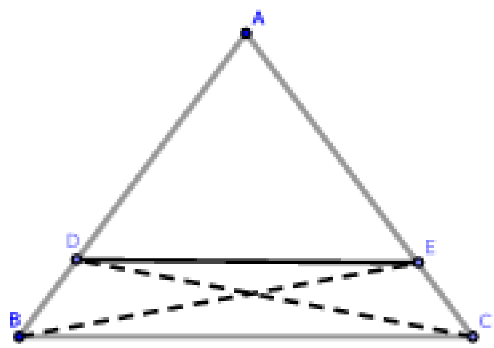

FIGURE 6.2.2. [6.2]

1. Suppose that $DE \parallel BC$ in $\triangle ABC$. We wish to show that $AD : DB :: AE : EC$.

Join BE, CD. The triangles $\triangle BDE$, $\triangle CED$ are on the same base DE and between the same parallels BC, DE. By [1.37], they are equal in area, and, regarding the proportionality of the areas of the triangles, we have that $\triangle ADE : \triangle BDE :: \triangle ADE : \triangle CDE$ [5.7]. We also have that $\triangle ADE : \triangle BDE :: AD : DB$ and $\triangle ADE : \triangle CDE :: AE : EC$, both by [6.1]. It follows that $AD : DB :: AE : EC$.

2. Now suppose $AD : DB :: AE : EC$. We wish to show that $DE \parallel BC$.

Let the same construction be made as in part 1. Then we have that $AD : BD :: \triangle ADE : \triangle BDE$ and $AE : EC :: \triangle ADE : \triangle CDE$, both by [6.1]. By hypothesis, we also have that $AD : DB :: AE : EC$. Hence it follows that $\triangle ADE : \triangle BDE :: \triangle ADE : \triangle CDE$.

By [5.9], $\triangle BDE = \triangle CDE$. These triangles also stand on the same base DE as well as on the same side of DE. By [1.39], they stand between the same parallels, and so $DE \parallel BC$. □

Note: The segment DE may cut the sides AB, AC extended through points B or C or through the angle at A, but it is clear that a separate figure for each of these cases is unnecessary.

Exercise.

1. If two segments are cut by three or more parallels, the intercepts on one are proportional to the corresponding intercepts on the other.

PROPOSITION 6.3. *ANGLES AND PROPORTIONALITY OF TRIANGLES. If a line bisects any angle of a triangle, it divides the opposite side into segments proportional to the adjacent sides. Conversely, if the segments into which a line constructed from any angle of a triangle divides the opposite side is proportional to the adjacent sides, that line bisects the angle.*

PROOF. If a line (AD) bisects any angle ($\angle BAC$) of a triangle ($\triangle ABC$), it divides the opposite side (BC) into segments (BD, DC) proportional to the adjacent sides (BA, AC). Conversely, if the segments (BD, DC) into which a line (AD) constructed from any angle ($\angle BAC$) of a triangle divides the opposite side is proportional to the adjacent sides (BA, AC) that line bisects the angle ($\angle BAC$).

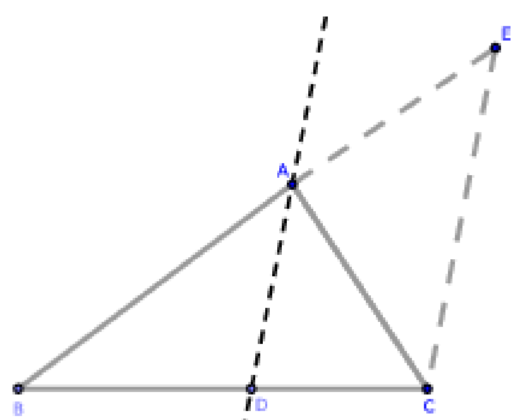

FIGURE 6.2.3. [6.3]

We prove each claim separately:

1. Suppose that AD bisects $\angle BAC$ of a triangle $\triangle ABC$. Through C, construct segment $CE \parallel AD$ to meet BA when BA is extended to the point E. Because BA meets the parallels AD, EC, we have that $\angle BAD = \angle AEC$ [1.29]. Also because AC meets the parallels AD, EC, we have that $\angle DAC = \angle ACE$.

By hypothesis, we also have that $\angle BAD = \angle DAC$. Therefore, $\angle ACE = \angle AEC$, and so $AE = AC$ [1.6].

Again, because $AD \parallel EC$, where EC is one of the sides of the triangle $\triangle BEC$, we have that $BD : DC :: BA : AE$ [6.2]. Since $AE = AC$ by the above, it follows that $BD : DC :: BA : AC$.

2. Now suppose that $BD : DC :: BA : AC$. We wish to prove that $\angle BAC$ is bisected.

Let the same construction be made as in part 1. Because $AD \parallel EC$, $BA : AE :: BD : DC$ [6.2]. But $BD : DC :: BA : AC$ by hypothesis. By [5.11], it follows that $BA : AE :: BA : AC$, and hence $AE = AC$ [5.9]. Therefore, $\angle AEC = \angle ACE$; we also have that $\angle ACE = \angle BAD$ [1.29] and that $\angle ACE = \angle DAC$. Hence $\angle BAD = \angle DAC$, and so the line AD bisects the angle $\angle BAC$. □

COROLLARY. *1. [6.3] holds when the line AD is replaced by a ray or segment of appropriate length, mutatis mutandis.*

Exercises.

1. If the segment AD bisects the external vertical angle $\angle CAE$, then $BA : AC :: BD : DC$, and conversely.

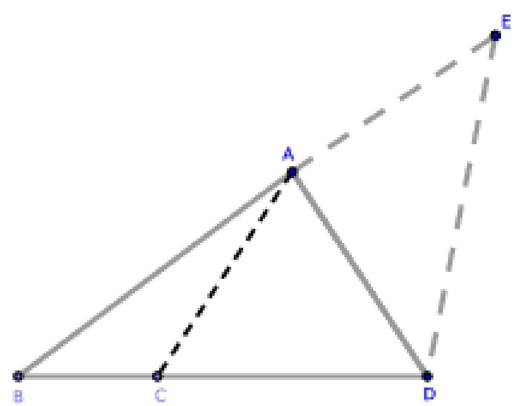

FIGURE 6.2.4. [6.3], #1

Cut off $AE = AC$. Join ED. Then the triangles $\triangle ACD$, $\triangle AED$ are evidently congruent; therefore the angle $\angle EDB$ is bisected, and hence $BA : AE :: BD : DE$ and $BA : AC :: BD : DC$ [6.3].

2. #1 has been proved by quoting [6.3]. Prove it independently, and prove [6.3] as an inference from it.

3. The internal and the external bisectors of the vertical angle of a triangle divide the base harmonically.

4. Any segment intersecting the legs of any angle is cut harmonically by the internal and external bisectors of the angle.

5. Any segment intersecting the legs of a right angle is cut harmonically by any two lines through its vertex which make equal angles with either of its sides.

6. If the base of a triangle is given in magnitude and position and if the ratio of the sides is also given, then the locus of the vertex is a circle which divides the base harmonically in the ratio of the sides.

7. If a, b, c denote the sides of a triangle $\triangle ABC$, and D, D' are the points where the internal and external bisectors of A meet BC, then prove that $DD' = \frac{2abc}{b^2-c^2}$.

8. In the same case as #7, if E, E', F, F' are points similarly determined on the sides CA, AB, respectively, prove that

$$\begin{aligned} \frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} &= 0 \\ \frac{a^2}{DD'} + \frac{b^2}{EE'} + \frac{c^2}{FF'} &= 0 \end{aligned}$$

PROPOSITION 6.4. *EQUIANGULAR TRIANGLES I. The sides about the equal angles of equiangular triangles are proportional, and those which stand opposite to the equal angles are homologous.*

PROOF. The sides about the equal angles of equiangular triangles ($\triangle BAC$, $\triangle CDE$) are proportional, and those which are opposite to the equal angles are homologous.

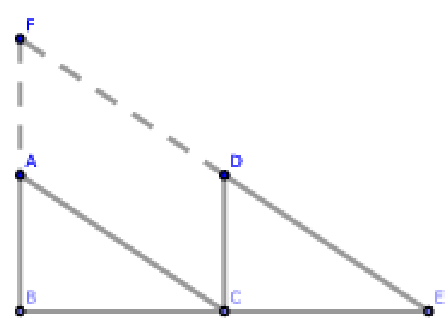

FIGURE 6.2.5. [6.4]

Let the sides BC, CE which stand opposite to the equal angles $\angle BAC$ and $\angle CDE$ be constructed so as to form one continuous segment and where the triangles stand on the same side of the segment such that the equal angles $\angle BCA$, $\angle CED$ do not share a common vertex. The sum $\angle ABC + \angle BCA$ is less than two right angles, but $\angle BCA = \angle BED$ by hypothesis. Therefore the sum $\angle ABE + \angle BED$ is less than two right angles, and so the segments AB, ED will meet if extended. Let them meet at F. Again, because $\angle BCA = \angle BEF$, we have that $CA \parallel EF$ [1.28].

Similarly, $BF \parallel CD$. Therefore, the figure $\boxdot ACDF$ is a parallelogram, and so $AC = DF$ and $CD = AF$. Because $AC \parallel FE$, $BA : AF :: BC : CE$ [6.2]. But $AF = CD$, therefore $BA : CD :: BC : CE$, and so by [5.16], we obtain that $BA : BC :: CD : CE$.

Again, because $CD \parallel BF$, we obtain that $BC : CE :: FD : DE$. But $FD = AC$, and so $BC : CE :: AC : DE$. And by [5.16], $BC : AC :: CE : DE$.

Since we have that $BA : BC :: CD : CE$ and $BC : CA :: CE : DE$, it follows that $BA : CA :: CD : DE$, and so the sides about the equal angles are proportional. □

This proposition may also be proved very simply by superposition by Fig. 6.2.2. Construct the two triangles be $\triangle ABC$, $\triangle ADE$ and let the second triangle $\triangle ADE$ be constructed to be placed on $\triangle ABC$ so that its two sides AD, AE fall on the sides AB, AC. Since $\angle ADE = \angle ABC$, $DE \parallel BC$. Hence by [6.2], $AD : DB :: AE : EC$, and so we have $AD : AB :: AE : AC$ and $AD : AE :: AB : AC$ by [5.16]. Therefore, the sides about the equal angles $\angle BAC$, $\angle DAE$ are proportional, and an analogous result follows for the others.

It can be shown by this proposition that two lines which meet at infinity are parallel. Let I denote the point at infinity through which the two given lines pass, and construct any two parallels intersecting them in the points A, B and A', B'. Then the triangles $\triangle AIB$, $\triangle A'IB'$ are equiangular. Therefore, $AI : AB :: A'I : A'B'$ where the first term of the proportion is equal to the third. By [5.14], the second term AB is equal to the fourth $A'B'$, and, being parallel to it, the lines AA', BB' are parallel [1.43].

Exercises.

1. If two circles intercept equal chords AB, $A'B'$ on any secant, the tangents AT, $A'T$ to the circles at the points of intersection are to one another as the radii of the circles.

2. If two circles intercept on any secant chords that have a given ratio, the tangents to the circles at the points of intersection have a given ratio, namely, the ratio compounded of the direct ratio of the radii and the inverse ratio of the chords.

3. Being given a circle and a line, prove that a point may be found such that the rectangle of the perpendiculars falling on the line from the points of intersection of the circle with any chord through the point shall be given.

4. If AB is the diameter of a semicircle ADB and $CD \perp AB$, construct through A a chord AF of the semicircle meeting CD at E such that the ratio $CE : EF$ may be given.

PROPOSITION 6.5. *EQUIANGULAR TRIANGLES II. If two triangles have proportional sides, they are equiangular, and the angles which are equal stand opposite the homologous sides.*

PROOF. If two triangles ($\triangle ABC$, $\triangle DEF$) have their sides proportional ($BA : AC :: ED : DF$; $AC : CB :: DF : FE$), then they are equiangular and the equal angles stand opposite the homologous sides.

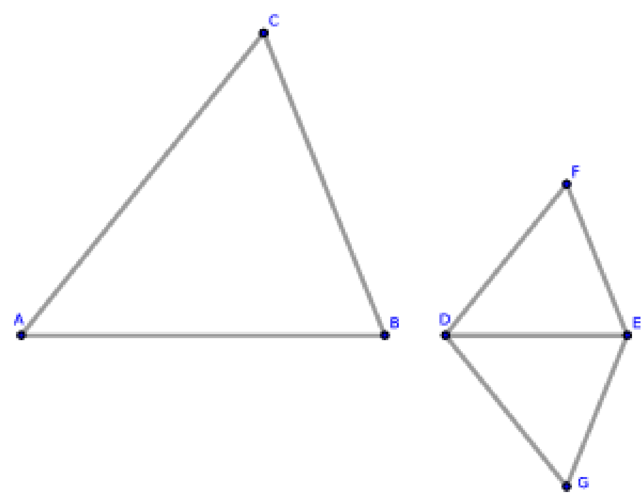

FIGURE 6.2.6. [6.5]

At the points D, E construct the angles $\angle EDG$, $\angle DEG$ equal to the angles $\angle BAC$, $\angle ABC$ of the triangle $\triangle ABC$. By [1.32], the triangles $\triangle ABC$, $\triangle DEG$ are equiangular. Therefore $BA : AC :: ED : DG$ by [1.4] and $BA : AC :: ED : DF$ by hypothesis. It follows that $DG = DF$.

Similarly, we can show that $EG = EF$. Hence the triangles $\triangle EDF$, $\triangle EDG$ have the sides ED, DF in one equal to the sides ED, DG in the other and the base EF equal to the base EG. By [1.8], they are equiangular. But the triangle $\triangle DEG$ is equiangular to $\triangle ABC$. Therefore the triangle $\triangle DEF$ is equiangular to $\triangle ABC$. □

Observation: In [Def. 6.1], two conditions are laid down as necessary for the similitude of polygons:

(a) The equality of angles;

(b) The proportionality of sides.

Now by [6.4] and [6.5], we see that if two triangles possess either condition, they also possess the other. Triangles are unique in this respect. In all other polygons, one of these conditions may exist without the other. Thus two quadrilaterals may have their sides proportional without having equal angles, or vice verse.

PROPOSITION 6.6. *If two triangles have one angle in one triangle equal to one angle in the other triangle and the sides about these angles are proportional, then the triangles are equiangular and have those angles equal which stand opposite to the homologous sides.*

PROOF. If two triangles ($\triangle ABC$, $\triangle DEF$) have one angle ($\angle BAC$) in one equal to one angle ($\angle EDF$) in the other, and the sides about these angles proportional ($BA : AC :: ED : DF$), then the triangles are equiangular and have those angles equal which stand opposite to the homologous sides.

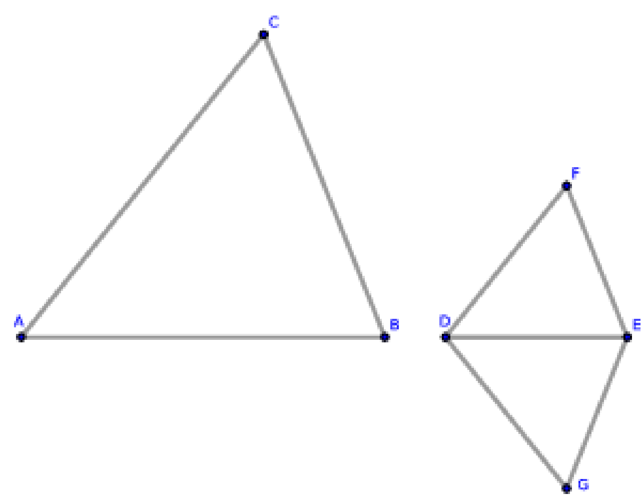

FIGURE 6.2.7. [6.6]

We recreate the construction as in [6.5]. Therefore $BA : AC :: ED : DG$ by [6.4] , $BA : AC :: ED : DF$ by hypothesis, and $DG = DF$.

Because $\angle EDG = \angle BAC$ by construction and $\angle BAC = \angle EDF$ by hypothesis, we have that $\angle EDG = \angle EDF$. Given that $DG = DF$ with DE in common, the triangles $\triangle EDG$ and $\triangle EDF$ are equiangular. But $\triangle EDG$ is equiangular to $\triangle BAC$, and so $\triangle EDF$ is equiangular to $\triangle BAC$. □

Note: as in the case of [6.4], an immediate proof of this proposition can also be obtained from [6.2].

COROLLARY. *1. If the ratio of two sides of a triangle are given as well as the angle between them, then the triangle is given in species.*

PROPOSITION 6.7. *EQUIANGULAR TRIANGLES III. If two triangles each have one angle equal to one angle in the other, if the sides about two other angles are proportional, and if the remaining angles of the same species (i. e. either both acute or both not acute), then the triangles are similar.*

PROOF. If two triangles ($\triangle ABC$, $\triangle DEF$) each have one angle equal to one angle ($\angle BAC = \angle EDF$) in the other, the sides about two other angles (B, E) are proportional ($AB : BC :: DE : EF$), and the remaining angles ($\angle BCA$, $\angle EFD$) of the same species (i. e. either both acute or both not acute), then the triangles are similar ($\triangle ABC \sim \triangle DEF$).

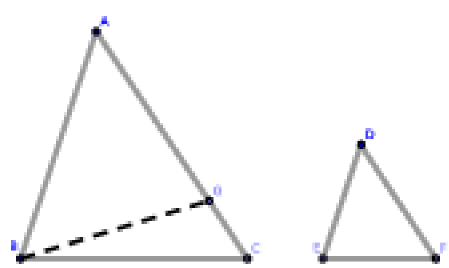

FIGURE 6.2.8. [6.7]

If the angles $\angle ABC$ and $\angle DEF$ are not equal, one must be greater than the other. Wlog, suppose $\angle ABC$ is the greater and that $\angle ABG = \angle DEF$. Then the triangles $\triangle ABG$, $\triangle DEF$ have two angles in one equal to two angles in the other and so are equiangular [1.32]. Therefore, $AB : BG :: DE : EF$ [6.4] and $AB : BC :: DE : EF$ by hypothesis. It follows that $BG = BC$. Hence $\angle BCG$, $\angle BGC$ must each be acute [1.17] and $\angle AGB$ must be obtuse. It follows that $\angle DFE = \angle AGB$ is obtuse. Since we have shown that $\angle ACB$ is acute, the angles $\angle ACB$, $\angle DFE$ are of different species; but by hypothesis, they are of the same species, a contradiction. Hence the angles $\angle CBA$ and $\angle FED$ are not unequal; that is, $\angle CBA = \angle FED$. Therefore, $\triangle ABG$, $\triangle DEF$ are equiangular, and so $\triangle ABG \sim \triangle DEF$. □

COROLLARY. *1. If two triangles $\triangle ABC$, $\triangle DEF$ have two sides in one proportional to two sides in the other, $AB : BC :: DE : EF$, and the angles at points A, D opposite one pair of homologous sides are equal, the angles at points C, F opposite the other are either equal or supplemental. This proposition is nearly identical with [6.7].*

COROLLARY. *2. If either of the angles at points C, F are right, the other angle must be right.*

PROPOSITION 6.8. *SIMILARITY OF RIGHT TRIANGLES. The triangles formed when a right triangle is divided by the perpendicular from the right angle to the hypotenuse are similar to the whole and to one another.*

PROOF. The triangles ($\triangle ACD$, $\triangle BCD$) formed when a right triangle ($\triangle ACB$) is divided by the perpendicular (CD) from the right angle ($\angle ACB$) to the hypotenuse are similar to the whole and to one another ($\triangle ACD \sim \triangle BCD$, $\triangle ACD \sim \triangle ACB$, $\triangle BCD \sim \triangle ACB$).

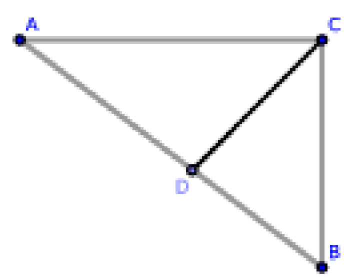

FIGURE 6.2.9. [6.8]

Since the two triangles $\triangle ACD$, $\triangle ACB$ have the angle $\angle BAC$ common, and since $\angle ADC = \angle ACB$ because each are right, the triangles are equiangular [1.32]. By [6.4], $\triangle ACD \sim \triangle ACB$. Similarly, it can be shown that $\triangle BCD \sim \triangle ACB$. It follows that $\triangle ACD \sim \triangle BCD$. □

COROLLARY. *1. The perpendicular CD is a mean proportional between the segments AD, DB of the hypotenuse. For since the triangles $\triangle ADC$, $\triangle CDB$ are equiangular, we have $AD : DC :: DC : DB$. Hence DC is a mean proportional between AD, DB (Def. 6.3).*

COROLLARY. *2. BC is a mean proportional between AB, BD, and AC is a mean proportional between AB, AD.*

COROLLARY. *3. The segments AD, DB are in the duplicate of $AC : CB$; in other words, $AD : DB :: AC^2 : CB^2$.*

COROLLARY. *4. $BA : AD$ are in the duplicate ratios of $BA : AC$, and $AB : BD$ are in the duplicate ratio of $AB : BC$.*

PROPOSITION 6.9. *From a given segment, we may cut off any required submultiple.*

PROOF. From a given segment (AB), we wish to cut off any required submultiple.

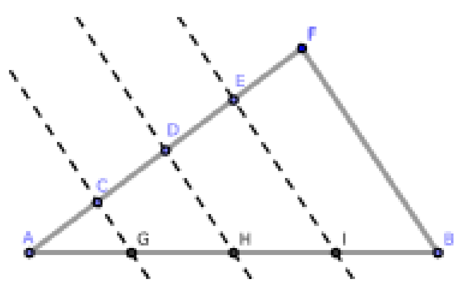

FIGURE 6.2.10. [6.9]

Suppose we wish to cut off a fourth. Construct the segment AF at any acute angle to AB (where AF is made sufficiently long). From AF, choose any point C and cut off the segments CD, DE, EF where each is equal to AC [1.3]. Join BF and construct $CG \parallel BF$. We claim that AG is a fourth of AB.

Since $CG \parallel BF$ where BF is the side of $\triangle ABF$, we have that $AC : AF :: AG : AB$ [6.2]. But AC is a fourth of AF by construction, and so AG is a fourth of AB. Since our choice of a fourth was arbitrary, any other required submultiple may similarly be cut off. □

Note: [1.10] is a particular case of this proposition.

PROPOSITION 6.10. *SIMILARLY DIVIDED SEGMENTS. We wish to divide a given undivided segment similarly to a given divided segment.*

PROOF. We wish to divide a given undivided segment (AB) similarly to a given divided segment (CD).

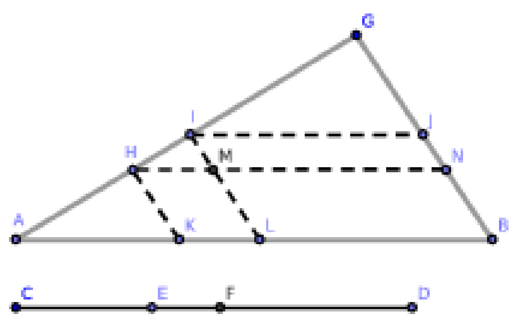

FIGURE 6.2.11. [6.10]

Construct AG at any acute angle with AB and cut off the parts AH, HI, IG respectively equal to the parts CE, EF, FD of the given divided segment CD. Join BG and construct HK, IL where each is parallel to BG. We claim that AB is divided similarly to CD.

Through H, construct $HN \parallel AB$, cutting IL at M. Now in the triangle $\triangle ALI$, we have that $HK \parallel IL$. By [6.2], $AK : KL :: AH : HI$, and by construction, we have that $AK : KL :: CE : EF$.

In $\triangle HNG$, we have that $MI \parallel NG$. By [6.2], $HM : MN :: HI : IG$. However, but by [1.34], $HM = KL$, $MN = LB$, and $HI = EF$, and by construction, $IG = FD$. Therefore, $KL : LB :: EF : FD$. Hence the segment AB is divided similarly to the segment CD. □

Exercises.

1. We wish to divide a given segment AB internally or externally in the ratio of two given lines, FG, HJ.

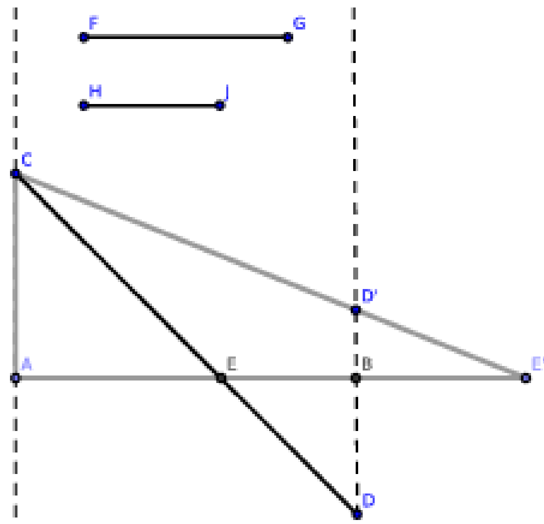

FIGURE 6.2.12. [6.11]

Through A and B construct any two parallels AC and BD in opposite directions. Cut off $AC = FG$, $BD = HJ$, and join CD; we claim that CD divides AB internally at E in the ratio of $FG : HJ$.

2. In #1, if BD' is constructed in the same direction with AC, then CD will cut AB externally at E in the ratio of $FG : HJ$.

COROLLARY. *1. The two points E, E' divide AB harmonically.*

This problem is manifestly equivalent to the following: given the sum or difference of two lines and their ratio, we wish to find the lines.

3. Any line AE' through the midpoint B of the base DD' of a triangle DCD' is cut harmonically by the sides of the triangle and a parallel to the base through the vertex.

4. Given the sum of the squares on two segments and their ratio, find the segments.

5. Given the difference of the squares on two segments and their ratio, find the segments.

6. Given the base and ratio of the sides of a triangle, construct it when any of the following data is given:

(a) the area;

(b) the difference on the squares of the sides;

(c) the sum of the squares on the sides;

(d) the vertical angle;

(e) the difference of the base angles.

PROPOSITION 6.11. *PROPORTIONAL SEGMENTS I. We wish to find a third proportional segment to two given segments.*

PROOF. We wish to find a third proportional segment to two given segments (JK, LM).

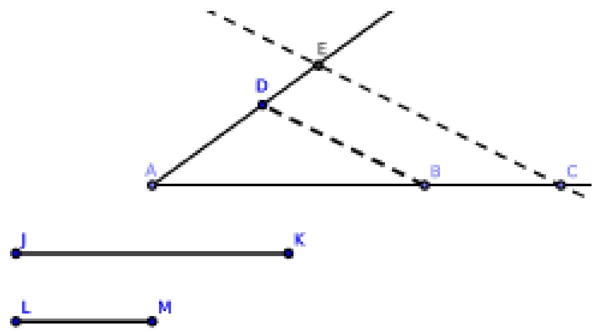

FIGURE 6.2.13. [6.11]

Construct any two segments AC, AE at an arbitrary acute angle. (The proof holds with rays or lines, *mutatis mutandis*.) Cut off $AB = JK$, $BC = LM$, and $AD = LM$. Join BD and construct $CE \parallel BD$. We claim that DE is the required third proportional segment.

In $\triangle CAE$, $BD \parallel CE$. Therefore $AB : BC :: AD : DE$ by [6.2]. But $AB = JK$ and $BC = LM = AD$. Therefore $JK : LM :: LM : DE$. Hence DE is a third proportional to JK and LM. □

COROLLARY. *5.11.1 Algebraically, this problem can be written as*

$$\begin{aligned} \frac{a}{b} &= \frac{b}{x} \\ &\Rightarrow \\ x &= \frac{b^2}{a} \end{aligned}$$

where a, b are fixed positive real numbers and x is a positive real variable.

Notes:

(1) Another solution can be inferred from [6.8]. For if AD, DC in that proposition are respectively equal to JK and LM, then DB will be the third proportional. Or again, if in Fig. 6.2.9, if $AD = JM$ and $AC = LM$, then AB will be the third proportional. Hence, we may infer a method of continuing the proportion to any number of terms.

Exercises.

1. If $\triangle AO\Omega$ is a triangle where the side $A\Omega$ is greater than AO, then if we cut off $AB = AO$, construct $BB \parallel AO$, cut off $BC = BB'$, and so on, the series of segments AB, BC, CD, etc., are in continual proportion.

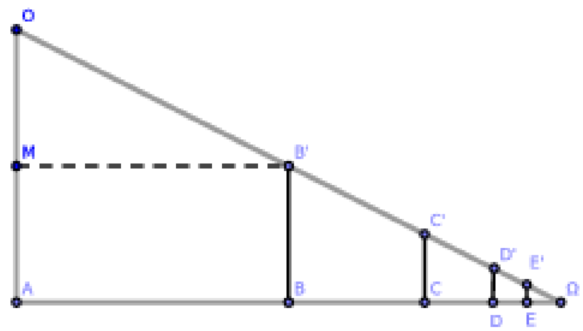

FIGURE 6.2.14. [6.2, #1]

2. $(AB–BC) : AB :: AB : A\Omega$. This is evident by constructing $MB' \parallel A\Omega$.

PROPOSITION 6.12. *PROPORTIONAL SEGMENTS II. We wish to find a fourth proportional to three given segments.*

PROOF. We wish to find a fourth proportional to three given segments (AK, BM, CP).

Construct any two segments DE, DF at an arbitrary acute angle. (The proof holds with rays or lines, *mutatis mutandis*.) Cut off $DG = AK$, $GE = BM$, and $DH = CP$. Join GH and construct $EF \parallel GH$ [1.31]. We claim that HF is the required fourth proportional segment.

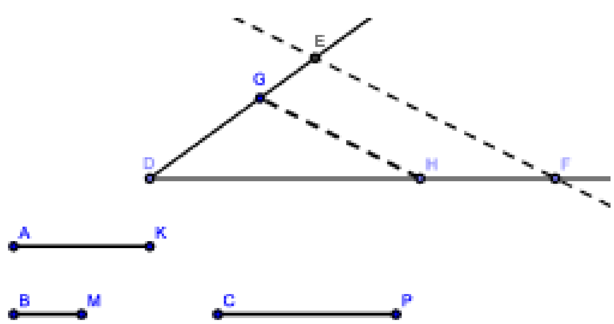

FIGURE 6.2.15. [6.12]

In $\triangle DEF$, $GH \parallel EF$. Therefore, $DG : GE :: DH : HF$ by [6.2]. But $DG = AK$, $GE = BM$, and $DH = CP$. Therefore $AK : BM :: CP : HF$, and so HF is a fourth proportional to AK, BM, and CP. □

Alternatively:

PROOF. Take two segments AD, BC which intersect at O.

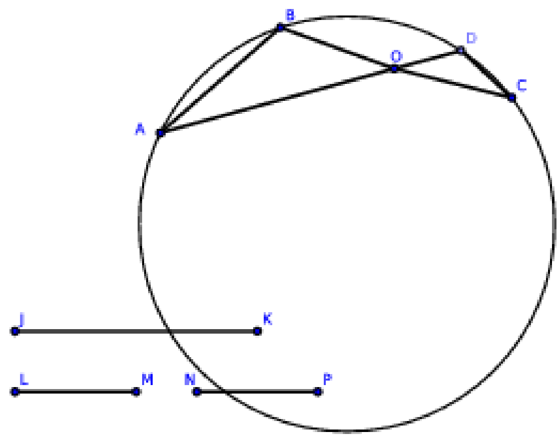

FIGURE 6.2.16. [6.12], Alternative proof

Construct $OA = JK$, $OB = LM$, $OC = MP$ and the circle $\circ ABC$ through the points A, B, C [4.5]. Extend AO through to the point D on the circumference of $\circ ABC$. We claim that OD is the fourth proportional required. The demonstration is evident from the similarity of the triangles $\triangle AOB$ and $\triangle COD$. □

COROLLARY. *5.12.1 Algebraically, this problem can be written as*

$$\begin{aligned} \frac{a}{b} &= \frac{c}{x} \\ &\Rightarrow \\ x &= \frac{bc}{a} \end{aligned}$$

where a, b, and c are fixed positive real numbers and x is a positive real variable.

PROPOSITION 6.13. *PROPORTIONAL SEGMENTS III. We wish to find a mean proportional between two given segments.*

PROOF. We wish to find a mean proportional between two given segments (EF, GH).

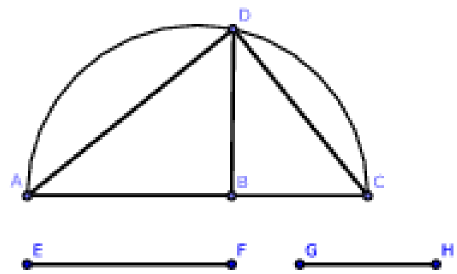

FIGURE 6.2.17. [6.13]

Construct segment AC such that $AC = EF + GH$ and cut parts AB, BC respectively equal to EF, GH. (The proof holds on rays or lines, *mutatis mutandis*.) On AC, construct a semicircle ADC. Construct $BD \perp AC$, meeting the semicircle at D. We claim that BD is the mean proportional required.

Join AD, DC. Since ADC is a semicircle, the angle $\angle ADC$ is right [3.31]. Hence, since $\triangle ADC$ is a right triangle and BD is a perpendicular from the right angle on the hypotenuse, BD is a mean proportional between AB, BC [6.8, Cor. 1]. That is, BD is a mean proportional between EF and GH (or $EF : BD :: BD : GH$). □

COROLLARY. *1. Algebraically, we have that*

$$\begin{aligned} \tfrac{a}{x} &= \tfrac{x}{b} \\ &\Rightarrow \\ x^2 &= ab \\ &\Rightarrow \\ x &= \sqrt{ab} \end{aligned}$$

where a, b are positive fixed real numbers and x is a positive real variable.

Exercises.

1. Another solution may be inferred from [6.8, Cor. 2].

2. If through any point within a circle a chord is constructed which is bisected at that point, its half is a mean proportional between the segments of any other chord passing through the same point.

3. The tangent to a circle from any external point is a mean proportional between the segments of any secant passing through the same point.

4. If through the midpoint C of any arc of a circle, a secant is constructed cutting the chord of the arc at D and the circle again at E, the chord of half the arc is a mean proportional between CD and CE.

5. If a circle is constructed touching another circle internally and with two parallel chords, the perpendicular from the center of the former on the diameter of the latter, which bisects the chords, is a mean proportional between the two extremes of the three segments into which the diameter is divided by the chords.

6. If a circle is constructed touching a semicircle and its diameter, the diameter of the circle is a harmonic mean between the segments into which the diameter of the semicircle is divided at the point of intersection.

7. State and prove the proposition corresponding to #5 for external contact of the circles.

PROPOSITION 6.14. *EQUIANGULAR PARALLELOGRAMS. We wish to prove that:*

1. Equiangular parallelograms ($\boxdot HACB$, $\boxdot CGDE$) which are equal in area have sides about the equal angles which are reciprocally proportional; that is, $AC : CE :: GC : CB$.

2. Equiangular parallelograms which have the sides about the equal angles reciprocally proportional are equal in area.

PROOF. We claim that:

1. Equiangular parallelograms ($\boxdot HACB$, $\boxdot CGDE$) which are equal in area have the sides about the equal angles reciprocally proportional; that is, $AC : CE :: GC : CB$.

2. Equiangular parallelograms which have the sides about the equal angles reciprocally proportional are equal in area.

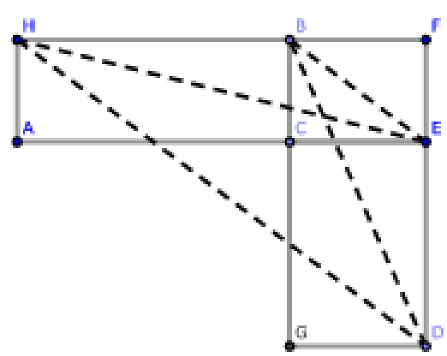

FIGURE 6.2.18. [6.14]

We prove each claim separately:

1. Let $\boxdot HACB$, $\boxdot CGDE$ be so placed as to form one segment AE such that the equal angles $\angle ACB$, $\angle ECG$ stand vertically opposite. Since $\angle ACB = \angle ECG$, add $\angle BCE$ to each, and we obtain the sum $\angle ACB + \angle BCE = \angle ECG + \angle BCE$. But $\angle ACB + \angle BCE$ equals two right angles [1.13], and so $\angle ECG + \angle BCE$ equals two right angles. By [1.14], BC, CG form one segment. We complete the parallelogram $\boxdot BCEF$.

Again, since the parallelograms $\boxdot HACB$, $\boxdot CGDE$ are equal in area by hypothesis, we have that

$$\begin{array}{rcll} \boxdot HACB : \boxdot BCEF & :: & \boxdot CGDE : \boxdot BCEF & [5.7] \\ AC : CE & :: & \boxdot HACB : \boxdot BCEF & [6.1] \\ \boxdot CGDE : \boxdot BCEF & :: & GC : CB & [6.1] \end{array}$$

Therefore, we have that $AC : CE :: GC : CB$; that is, the sides about the equal angles are reciprocally proportional.

2. Let $AC : CE :: GC : CB$. We wish to prove that the parallelograms $\boxdot HACB$, $\boxdot CGDE$ are equal in area.

Let the same construction be made as in part 1:

$$\begin{array}{rcll} AC : CE & :: & \boxdot HACB : \boxdot BCEF & [6.1] \\ \boxdot CGDE : \boxdot BCEF & :: & GC : CB & [6.1] \\ AC : CE & :: & GC : CB & \text{by hypothesis} \end{array}$$

Therefore, $\boxdot HACB : \boxdot BCEF :: \boxdot CGDE : \boxdot BCEF$, and hence $\boxdot HACB = \boxdot CGDE$ [5.9]. □

Alternatively:

PROOF. Join HE, BE, HD, BD. The area of the parallelogram $\boxdot HACB = 2 \cdot \triangle HBE$, and the area of the parallelogram $\boxdot CGDE = 2 \cdot \triangle BDE$. Therefore $\triangle HBE = \triangle BDE$, and by [1.39.], $HD \parallel BE$. Hence $HB : BF :: DE : EF$; that is, $AC : CE :: GC : CB$.

Part 2 may be proved by reversing this demonstration.

Another demonstration of this proposition may be obtained by extending the lines HA and DG to meet at I. Then by [1.43], the points I, C, F are collinear, and the proposition is evident. □

PROPOSITION 6.15. *EQUAL TRIANGLES.*

1. Two triangles equal in area which have one angle in one triangle equal to one angle in the other triangle have reciprocally proportional sides about these angles.

2. Two triangles which have one angle in one triangle equal to one angle in the other triangle and the sides about these angles reciprocally proportional are equal in area.

PROOF. We wish to prove that:

1. Two triangles equal in area ($\triangle ACB$, $\triangle DCE$) which have one angle ($\angle BCA$) in one equal to one angle ($\angle DCE$) in the other have reciprocally proportional sides about these angles.

2. Two triangles which have one angle in one triangle equal to one angle in the other triangle and the sides about these angles reciprocally proportional are equal in area.

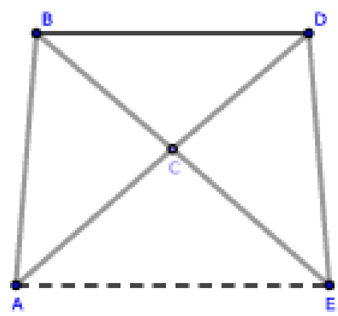

FIGURE 6.2.19. [6.15]

We prove each claim separately:

1. Let the equal angles be placed as to be vertically opposite such that AC, CD forms the segment AD; then it may be demonstrated, as in the previous proposition, that BC, CE form one segment. Join BD.

Since the triangles $\triangle ACB = \triangle DCE$, we have that

$$\begin{array}{lcll} \triangle ACB : \triangle BCD & :: & \triangle DCE : \triangle BCD & [5.7] \\ \triangle ACB : \triangle BCD & :: & AC : CD & [6.1] \\ \triangle DCE : \triangle BCD & :: & EC : CB & [6.1] \end{array}$$

Therefore, $AC : CD :: EC : CB$, or the sides about the equal angles are reciprocally proportional.

2. If $AC : CD :: EC : CB$, we wish to prove that $\triangle ACB = \triangle DCE$.

Let the same construction be made; then we have that

$$\begin{array}{lll} AC : CD & :: & \triangle ACB : \triangle BCD \quad [6.1] \\ EC : CB & :: & \triangle DCE : \triangle BCD \quad [6.1] \\ AC : CD & :: & EC : CB \quad \text{(by hypothesis)} \end{array}$$

Therefore, $\triangle ACB : \triangle BCD :: \triangle DCE : \triangle BCD$, and so $\triangle ACB = \triangle DCE$ [5.9]. □

This proposition might have been appended as a corollary to [6.14] since the triangles are the halves of equiangular parallelograms; it may also be proven by joining AE and showing that it is parallel to BD.

PROPOSITION 6.16. *PROPORTIONAL RECTANGLES. We wish to show that:*

1. If four segments are proportional, the rectangle contained by the extremes is equal in area to the rectangle contained by the means.

2. If the rectangle contained by the extremes of four segments is equal in area to the rectangle contained by the means, the four segments are proportional.

PROOF. We wish to show that:

1. If four segments (AB, CD, LM, NP) are proportional, the rectangle ($AB.NP$) contained by the extremes is equal in area to the rectangle ($CD.LM$) contained by the means.

2. If the rectangle contained by the extremes of segments are equal to the rectangle contained by the means, the four lines are proportional.

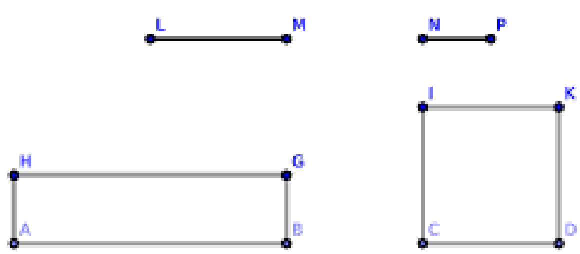

FIGURE 6.2.20. [6.16]

We solve each claim separately:

1. Suppose that the segments AB, CD, LM, NP are proportional (that is, $AB : CD :: LM : NP$). Construct $AH = NP$ and $CI = LM$ at right angles

to AB and CD, respectively, and complete the rectangles. Because $AB : CD :: LM : NP$ by hypothesis, we have that $AB : CD :: CI : AH$.

Notice that the parallelograms $\boxdot HABG$, $\boxdot ICDK$ are equiangular and the sides about their equal angles are reciprocally proportional. By [6.14], they are equal in area. Since since $AH = NP$, we have that $\boxdot HABG = AB.NP$.

Similarly, $\boxdot ICDK = CD.LM$. It follows that $AB.NP = CD.LM$, or the rectangle contained by the extremes is equal to the rectangle contained by the means.

2. Suppose that $AB.NP = CD.LM$. We now wish to prove $AB : CD :: LM : NP$.

We carry out the same construction as in part 1: because $AB.NP = CD.LM$, $AH = NP$, and $CI = LM$, we have that $\boxdot HABG = \boxdot ICDK$. Since these parallelograms are equiangular, the sides about their equal angles are reciprocally proportional. Therefore $AB : CD :: CI : AH$, or $AB : CD :: LM : NP$. □

Alternatively:

PROOF. Place the four segments in a concurrent position so that the extremes form one continuous segment and the means form a second continuous segment.

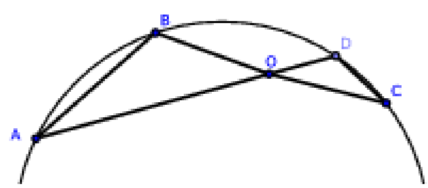

FIGURE 6.2.21. [6.16], Alternative proof

Place the four segments in the order AO, BO, OD, OC. Join AB, CD. Because $AO : OB :: OD : OC$ and $\angle AOB = \angle DOC$, the triangles $\triangle AOB$, $\triangle COD$ are equiangular. Hence, the four points A, B, C, D are concyclic, and so by [3.35], $AO.OC = BO.OD$. □

COROLLARY. *6.16.1 Algebraically, the above result states in part that if*

$$\frac{a}{b} = \frac{c}{d}$$

then

$$ad = bc$$

PROPOSITION 6.17. *LINES AND RECTANGLES.*

1. If three segments are proportional, the rectangle contained by the endpoints is equal in area to the square of the mean.

2. If the rectangle contained by the endpoints of three segments is equal in area to the square of the mean, then the three segments are proportional.

PROOF. We wish to show that:

1. If three segments (AB, CD, GH) are proportional ($AB : CD :: CD : GH$), then the rectangle ($AB.GH$) contained by the extremes is equal in area to the square of the mean (CD^2).

2. If the rectangle contained by the endpoints of three segments are equal in area to the square of the mean, then the three lines are proportional.

FIGURE 6.2.22. [6.17]

We prove each claim separately:

1. Suppose that $AB : CD :: CD : GH$ and that $CD = EF$. By hypothesis, we have that $AB : CD :: EF : GH$. By [6.16], $AB.GH = CD.EF$. But $CD.EF = CD^2$. Therefore, $AB.GH = CD^2$; that is, the rectangle contained by the extremes is equal to the square of the mean.

2. Now suppose that $AB.GH = CD^2$. Under the same construction, since $AB.GH = CD.EF$, we have that $AB : CD :: EF : GH$ where $CD = EF$. It follows that $AB : CD :: CD : GH$; that is, the three segments are proportionals.

□

Note: This proposition may also be inferred as a corollary to [6.16].

Exercises.

1. Construct a Corollary similar to Corollary 6.16.1 which states the results of [6.17] algebraically.

2. If a segment CD bisects a vertical angle at a point C of any triangle $\triangle ACB$, its square added to the rectangle $AD.DB$ contained by the segments of the base is equal to the rectangle contained by the sides.

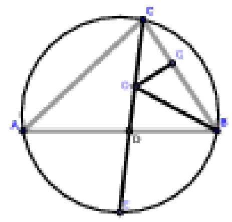

FIGURE 6.2.23. [6.17, #1]

Construct a circle about the triangle and extend CD to meet it at E. Clearly, the triangles $\triangle ACD$, $\triangle ECB$ are equiangular. By [6.4], $AC : CD :: CE : CB$, and by [3.35], we obtain

$$AC.CB = CE.CD = CD^2 + CD.DE = CD^2 + AD.DB$$

3. If the segment CD bisects the external vertical angle of any triangle $\triangle ACB$, its square subtracted from the rectangle $AD.DB$ is equal to $AC.CB$. (See Fig. 6.2.23).

4. The rectangle contained by the diameter of the circumscribed circle, and the radius of the inscribed circle of any triangle, is equal to the rectangle contained by the segments of any chord of the circumscribed circle passing through the center of the inscribed. (See Fig. 6.2.23).

Let O be the center of the inscribed circle. Join OB, construct the perpendicular OG, and construct construct the diameter EF of the circumscribed circle. Now we have that $\angle ABE = \angle ECB$ and $\angle ABO = \angle OBC$ [3.27]; therefore $\angle EBO =$ sum of$\angle OCB$, $\angle OBC = \angle EOB$. Hence, $EB = EO$. Again, the triangles $\triangle EBF$, $\triangle OGC$ are equiangular because $\angle EFB = \angle ECB$ and $\angle EBF = \angle OGC$ (since each are right). Therefore, $EF : EB :: OC : OG$, from which it follows that

$$EF.OG = EB.OC = EO.OC$$

5. #3 may be extended to each of the escribed circles of $\triangle ACB$.

6. The rectangle contained by two sides of a triangle is equal to the rectangle contained by the perpendicular and the diameter of the circumscribed circle. (See Fig. 6.2.24).

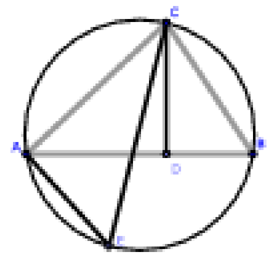

FIGURE 6.2.24. [6.17, #5]

Let CE be the diameter and join AE. Then the triangles $\triangle ACE$, $\triangle DCB$ are equiangular; hence $AC : CE :: CD : CB$, and therefore $AC.CB = CD.CE$.

7. If a circle passing through the angle at point A of a parallelogram $\boxdot ABCD$ intersects the two sides AB, AD again at the points E, G and the diagonal AC again at F, then $AB.AE + AD.AG = AC.AF$. (See Fig. 6.2.25).

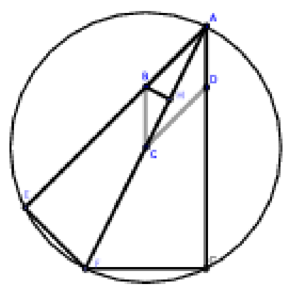

FIGURE 6.2.25. [6.17, #6]

Join EF, FG, and make the angle $\angle ABH = \angle AFE$. Then the triangles $\triangle ABH$, $\triangle AFE$ are equiangular. Therefore $AB : AH :: AF : AE$. Hence, $AB.AE = AF.AH$. Again, it is clear that the triangles $\triangle BCH$, $\triangle GAF$ are equiangular, and therefore $BC : CH :: AF : AG$. Hence $BC.AG = AF.CH$, or $AD.AG = AF.CH$; but we have proved $AB.AE = AF.AH$. Therefore $AD.AG + AB.AE = AF.AC$.

8. If DE, DF are parallels to the sides of a triangle $\triangle ABC$ from any point D in the base, then $AB.AE + AC.AF = AD^2 + BD.DC$. Hint: deduce this from #6.

9. If through a point O within a triangle $\triangle ABC$ parallels EF, GH, IK are constructed to the sides, the sum of the rectangles of their segments is equal to the rectangle contained by the segments of any chord of the circumscribing circle passing through O.

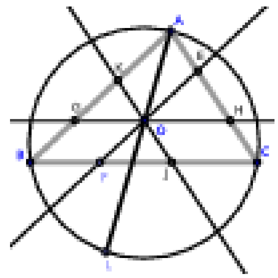

FIGURE 6.2.26. [6.17, #8]

Notice that

$$AO.AL = AB.AK + AC.AE$$

and

$$AO^2 = AG.AK + AH.AE – GO.OH$$

Hence,

$$AO.OL = BG.AK + CH.AE + GO.OH$$

or

$$AO.OL = EO.OF + IO.OK + GO.OH$$

10. The rectangle contained by the side of an inscribed square standing on the base of a triangle and the sum of the base and altitude is equal to twice the area of the triangle.

11 The rectangle contained by the side of an escribed square standing on the base of a triangle and the difference between the base and altitude is equal to twice the area of the triangle.

12. If from any point P in the circumference of a circle a perpendicular is constructed to any chord, its square is equal to the area of the rectangle contained by the perpendiculars from the endpoints of the chord on the tangent at P.

13. If O is the point of intersection of the diagonals of a cyclic quadrilateral $ABCD$, the areas of the four rectangles $AB.BC$, $BD.CD$, $CD.DA$, $DA.AB$ are proportional to lengths of the four segments BO, CO, DO, AO.

14. Ptolemy's Theorem: the sum of the rectangles of the opposite sides of a cyclic quadrilateral $ABCD$ is equal in area to the rectangle contained by its diagonals.

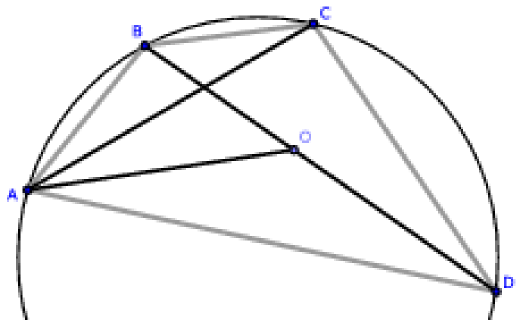

FIGURE 6.2.27. [6.17, #13]

Construct the circle $\circ ABC$ on quadrilateral $ABCD$. $\angle DAO = \angle CAB$. Then the triangles $\triangle DAO$, $\triangle CAB$ are equiangular. Therefore, $AD : DO :: AC : CB$, and it follows that $AD.BC = AC.DO$. Again, the triangles $\triangle DAC$, $\triangle OAB$ are equiangular and $CD : AC :: BO : AB$. Therefore $AC.CD = AC.BO$, and so $AD.BC + AB.CD = AC.BD$.

15. If the quadrilateral $ABCD$ is not cyclic, prove that the three rectangles $AB.CD$, $BC.AD$, $AC.BD$ are proportional to the three sides of a triangle which has an angle equal to the sum of a pair of opposite angles of the quadrilateral.

16. Prove by using [6.11] that if perpendiculars fall on the sides and diagonals of a cyclic quadrilateral, from any point in the circumference of the circumscribed circle, the rectangle contained by the perpendiculars on the diagonals is equal to the rectangle contained by the perpendiculars on either pair of opposite sides.

17. If AB is the diameter of a semicircle and PA, PB are chords from any point P in the circumference, and if a perpendicular to AB from any point C meets PA, PB at D and E and the semicircle at point F, then CF is a mean proportional between CD and CE.

PROPOSITION 6.18. *CONSTRUCTION OF A SIMILAR POLYGON. We wish to construct a polygon on a given segment which is similar to a given polygon and similarly placed.*

PROOF. Let polygon $CDEFG$ and segment AB be given. We wish to construct a polygon on AB which is similar to polygon $CDEFG$ and which is similarly placed.

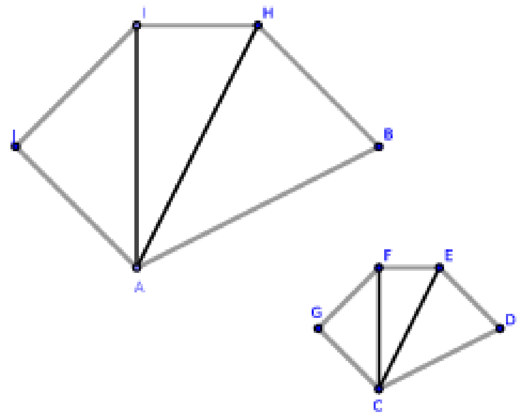

FIGURE 6.2.28. [6.18]

Join CE, CF, and construct triangle $\triangle ABH$ on AB equiangular to $\triangle CDE$ and similarly placed as regards CD; that is, construct $\angle ABH = \angle CDE$ and $\angle BAH = \angle DCE$. Also construct the triangle $\triangle HAI$ equiangular to $\triangle ECF$ and similarly placed. Finally, construct the triangle $\triangle IAJ$ equiangular and similarly placed with $\triangle FCG$. We claim that $ABHIJ$ is the required polygon.

By construction, it is evident that the figures are equiangular, and it is only required to prove that the sides about the equal angles are proportional.

Because the triangle $\triangle ABH$ is equiangular to $\triangle CDE$, we have that $AB : BH :: CD : DE$ [6.4]. Hence the sides about the equal angles at points B and D are proportional. Again from the same triangles, we have $BH : HA :: DE : EC$, and from the triangles $\triangle IHA$, $\triangle FEC$, we have $HA : HI :: EC : EF$. Therefore, $BH : HI :: DE : EF$, or the sides about the equal angles $\angle BHI$, $\angle DEF$ are proportional. This result follows about the other equal angles, *mutatis mutandis*. Hence by [Def. 6.1], the figures are similar. □

Observation: in the foregoing construction, the segment AB is homologous to CD, and it is evident that we may take AB to be homologous to any other side of the given figure $CDEFG$. Again, in each case, if the figure $ABHIJ$ is turned round the segment AB until it falls on the other side, it will still be similar to the figure $CDEFG$. Hence on a given line AB, there can be constructed two figures each similar to a given figure $CDEFG$ and having the given segment AB homologous to any given side CD of the given figure.

The first of the figures thus constructed is said to be directly similar, and the second is said to be inversely similar to the given figure.

COROLLARY. *1. Twice as many polygons may be constructed on AB similar to a given polygon $CDEFG$ as that figure has sides.*

COROLLARY. *2. If the figure $ABHIJ$ is applied to $CDEFG$ so that the point A coincides with C and that the line AB is placed along CD, then the points H, I, J will be respectively on the segments CE, CF, CG. Also, the sides BH, HI, IJ of the one polygon will be respectively parallel to their homologous sides DE, EF, FG of the other.*

COROLLARY. *3. If segments constructed from any point O in the plane of a figure to all its angular points are divided in the same ratio, the lines joining the points of division will form a new figure similar to and having every side parallel to the homologous side of the original.*

Note: [6.19] is the first of Euclid's Proposition in which the technical term "duplicate ratio" occurs. Most students find it difficult to understand either Euclid's proof or his definition. Due to this, we follow Euclid's proof with John Casey's alternative proof which makes use of a new definition of the duplicate ratio of two lines: *the ratio of the squares constructed on these segments*.

PROPOSITION 6.19. *RATIOS OF SIMILAR TRIANGLES. Similar triangles have their areas to one another in the duplicate ratio of their homologous sides.*

PROOF. We claim that similar triangles ($\triangle ABC$, $\triangle DEF$) have their areas to one another in the duplicate ratio of their homologous sides.

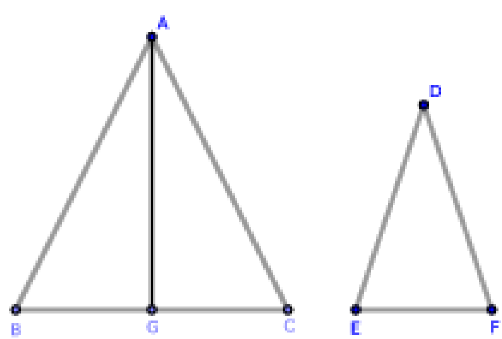

FIGURE 6.2.29. [6.19] (α)

Take BG as a third proportional to BC, EF [6.11]; that is, $BC : BG :: BG : EF$ Join AG. Then because $\triangle ABC \sim \triangle DEF$, $AB : BC :: DE : EF$; hence we also have that $AB : DE :: BC : EF$ [5.16]. Since we also have that $BC : EF :: EF : BG$, by [5.11] we also have that $AB : DE :: EF : BG$. Hence the sides of the triangles $\triangle ABG$, $\triangle DEF$ about the equal angles at points B, E are reciprocally proportional, and therefore $\triangle ABG$, $\triangle DEF$ are equal in area.

Again, since the segments BC, EF, BG are continual proportionals, $BC : BG$ is in the duplicate ratio of $BC : EF$ [Def. 5.10]; but $BC : BG :: \triangle ABC : \triangle ABG$. Therefore $\triangle ABC : \triangle ABG$ in the duplicate ratio of $BC : EF$. But we have shown that the triangle $\triangle ABG = \triangle DEF$. Therefore, the triangle $\triangle ABC$ is to the triangle $\triangle DEF$ in the duplicate ratio of $BC : EF$. □

Casey's proof:

PROOF. Suppose that $\triangle ABC \sim \triangle DEF$. On AB and DE, construct squares ($\boxdot AGHB$ and $\boxdot DLME$, respectively), and through points C and F construct segments parallel to AB and DE. Extend AG, BH, DL, and EM to points J, I, O, and N, respectively; this constructs the rectangles $\boxdot JABI$ and $\boxdot ODEN$.

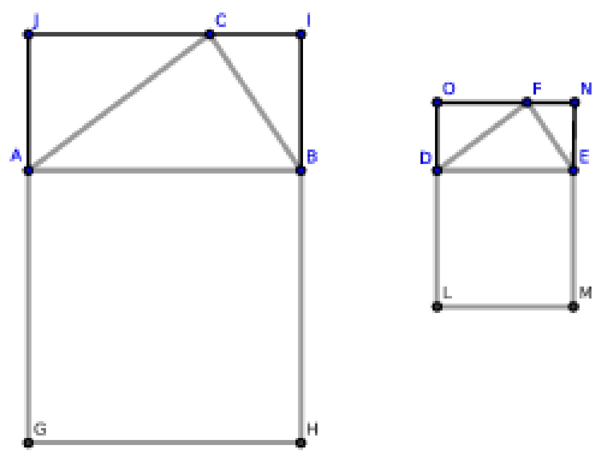

FIGURE 6.2.30. [6.19] (β)

Clearly, the triangles $\triangle JAC$, $\triangle ODF$ are equiangular.

Hence:

$$\begin{array}{lcll} JA : AC & :: & OD : DF & \text{[6.4] and} \\ AC : AB & :: & DF : DE & \text{[6.4]} \Rightarrow \\ JA : AB & :: & OD : DE & \end{array}$$

Since $AB = AG$ and $DE = DL$ by construction, we also have that $JA : AG :: OD : DL$.

Again,

$$\begin{array}{rcll} JA : AG & :: & \boxdot JABI : \boxdot AGHB & \text{[6.1] and} \\ OD : DL & :: & \boxdot ODEN : \boxdot DLME & \text{[6.1]} \Rightarrow \\ \boxdot JABI : \boxdot AGHB & :: & \boxdot ODEN : \boxdot DLME & \end{array}$$

Therefore, $\boxdot JABI : \boxdot ODEN :: \boxdot AGHB : \boxdot DLME$ by [5.16], and hence

$$\triangle ABC : \triangle DEF :: AB^2 : DE^2$$

□

Exercises.

1. If one of two similar triangles has a side that is 50% longer than the homologous sides of the other, what is the ratio of their areas?

2. When the inscribed and circumscribed regular polygons of any common number of sides to a circle have more than four sides, the difference of their areas is less than the square of the side of the inscribed polygon.

PROPOSITION 6.20. *DIVISION OF SIMILAR POLYGONS. Similar polygons may be divided:*

1. into the same number of similar triangles;

2. such that the corresponding triangles have the same ratio to one another which the polygons have;

3. such that the polygons are to each other in the duplicate ratio of their homologous sides.

PROOF. Let $ABHIJ$, $CDEFG$ be the polygons, and let the sides AB, CD be homologous. Join AH, AI, CE, CF. We prove each claim separately:

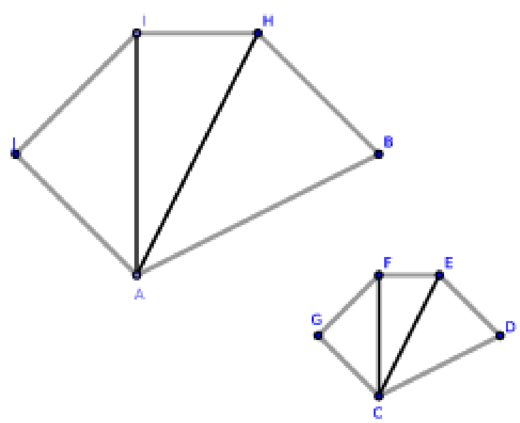

FIGURE 6.2.31. [6.20]

1. We claim that similar polygons may be divided into the same number of similar triangles.

Since the polygons are similar, they are equiangular and have proportional sides about their equal angles [Def. 6.1]. Hence, the angle at B is equal to the angle at D, and $AB : BH :: CD : DE$. By [6.6], the triangle $\triangle ABH$ is equiangular to $\triangle CDE$, and so $\angle BHA = \angle DEC$. But $\angle BHI = \angle DEF$ by hypothesis; therefore, $\angle AHI = \angle CEF$.

Again, because the polygons are similar, $IH : HB :: FE : ED$; and since $\triangle ABH \sim \triangle CDE$, $HB : HA :: ED : EC$. It follows that $IH : HA :: FE : EC$, and we have shown that $\angle IHA = \angle FEC$. Therefore, $\triangle IHA$, $\triangle FEC$ are equiangular. Similarly, it can be shown that the remaining triangles are equiangular.

2. We claim that similar polygons may be divided such that the corresponding triangles have the same ratio to one another which the polygons have.

Since $\triangle ABH \sim \triangle CDE$, we have that

$$\triangle ABH : \triangle CDE \text{ in the duplicate ratio of } AH : CE \text{ [6.19]}$$

Similarly,

$$\triangle AHI : \triangle CEF \text{ in the duplicate ratio of } AH : CE$$

Hence, $\triangle ABH : \triangle CDE = \triangle AHI : \triangle CEF$ [5.9]. Similarly, $AHI : CEF = AIJ : CFG$.

In these equal ratios, the triangles $\triangle ABH$, $\triangle AHI$, $\triangle AIJ$ are the antecedents, the triangles $\triangle CDE$, $\triangle CEF$, $\triangle CFG$ are the consequents, and any one of these equal ratios is equal to the ratio of the sum of all the antecedents to the sum of all the consequents [5.7]. Therefore,

$$\triangle ABH : \triangle CDE :: \text{polygon } ABHIJ : \text{ polygon } CDEFG$$

3. We claim that similar polygons are to each other in the duplicate ratio of their homologous sides.

The triangle

$$\triangle ABH : \triangle CDE \text{ in the duplicate ratio of } AB : CD \text{ [6.19]}$$

Hence by (2),

$$\text{polygon } ABHIJ : \text{polygon } CDEFG \text{ in the duplicate ratio of } AB : CD$$

□

COROLLARY. *1. The perimeters of similar polygons are to one another in the ratio of their homologous sides.*

COROLLARY. *2. As squares are to similar polygons, the duplicate ratio of two segments is equal to the ratio of the squares constructed on them.*

COROLLARY. *3. Similar portions of similar figures bear the same ratio to each other as the wholes of the figures.*

COROLLARY. *4. Similar portions of the perimeters of similar figures are to each other in the ratio of the whole perimeters.*

Exercises.

1. If two figures are similar, to each point in the plane of one there will be a corresponding point in the plane of the other.

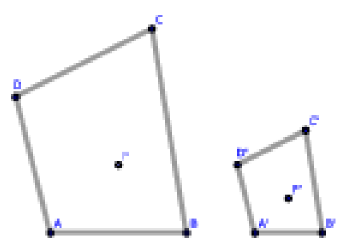

FIGURE 6.2.32. [6.20, #1]

Let $ABCD$, $A'B'C'D'$ be the two figures and P a point inside of $ABCD$. Join AP, BP, and construct a triangle $\triangle A'P'B'$ on $A'B'$ similar to $\triangle APB$; clearly, segments from P' to the angular points of $A'B'C'D'$ are proportional to the lines from P to the angular points of $ABCD$.

2. If two figures are directly similar and in the same plane, there is in the plane called a homologous point with respect to the other (which may be regarded as belonging to either figure).

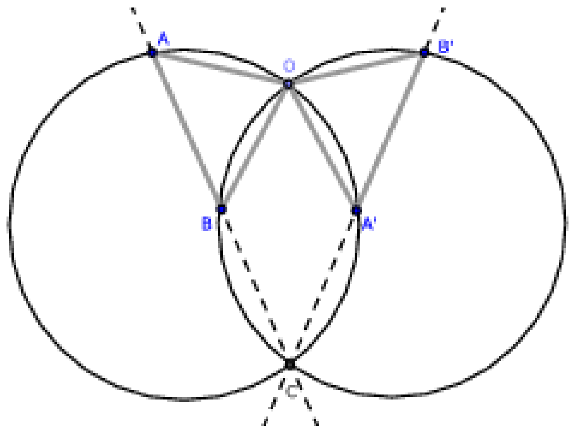

FIGURE 6.2.33. [6.20, #2]. See also [Def 6.8].

Let AB, $A'B'$ be two homologous sides of the figures and C their point of intersection. Through the two triads of points A, A', C and B, B', C construct two circles intersecting again at the point O: we claim that O is the required point. Clearly, $\triangle OAB \sim \triangle OAB$ and either may be turned round the point O, so that the two bases, AB, $A'B'$ will be parallel.

3. Two regular polygons of n sides each have n centers of similitude.

4. If any number of similar triangles have their corresponding vertices lying on three given lines, they have a common center of similitude.

5. If two figures are directly similar and have a pair of homologous sides parallel, every pair of homologous sides will be parallel.

Definition: Figures such as those in #5 are said to be *homothetic*.

6. If two figures are homothetic, the segments joining corresponding angular points are concurrent, and the point of concurrence is the center of similitude of the figures.

7. If two polygons are directly similar, either may be turned round their center of similitude until they become homothetic, and this may be done in two different ways.

8. Two circles are similar figures.

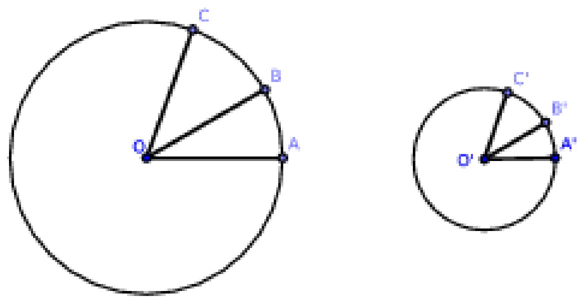

FIGURE 6.2.34. [6.20, #8]

Let O, O' be their centers, and let the angle $\angle AOB$ be made indefinitely small so that the arc AB may be regarded as a straight line; construct $\angle AOB = \angle A'O'B'$. We claim that the triangles $\triangle AOB \sim \triangle A'O'B'$.

Again, construct the angle $\angle BOC$ indefinitely small and set $\angle B'O'C' = \angle BOC$. Hence, $\triangle BOC \sim \triangle B'O'C'$. Proceeding in this way, we see that the circles can be divided into the same number of similar elementary triangles. Hence the circles are similar figures.

9. Sectors of circles having equal central angles are similar figures.

10. As any two points of two circles may be regarded as homologous, two circles have in consequence an infinite number of centers of similitude. Their locus is the circle, whose diameter is the line joining the two points for which the two circles are homothetic.

11. The areas of circles are to one another as the squares of their diameters. For they are to one another as the similar elementary triangles into which they are divided, and these are as the squares of the radii.

12. The circumferences of circles are proportional to their diameters (see [6.20, Cor. 1]).

13. The circumference of sectors having equal central angles are proportional to their radii. Hence if a, a' denote the arcs of two sectors which stand opposite equal angles at the centers, and if r, r' are their radii, then we have that $a/r = a'/r'$.

14. The area of a circle is equal to half the rectangle contained by the circumference and the radius. (This is evident by dividing the circle into elementary triangles, as in #8.)

15. The area of a sector of a circle is equal to half the rectangle contained by the arc of the sector and the radius of the circle.

PROPOSITION 6.21. *TRANSITIVITY OF SIMILAR POLYGONS. Polygons which are similar to the same figure are similar to one another.*

PROOF. We claim that polygons (ABC, DEF) which are similar to the same figure (GHI) are similar to one another (that is, the property of similarity is transitive).

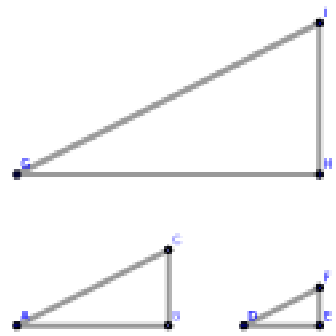

FIGURE 6.2.35. [6.21] Note that the polygons need not be triangles.

Since $ABC \sim GHI$, they are equiangular and have the sides about their equal angles proportional. Similarly, DEF and GHI are equiangular and have the sides about their equal angles proportional. Hence ABC and DEF are equiangular and have the sides about their equal angles proportional; or, $ABC \sim DEF$. □

COROLLARY. *1. Two similar polygons which are homothetic to a third are homothetic to one another.*

Exercise.

1. If three similar polygons are respectively homothetic, then their three centers of similitudes are collinear.

PROPOSITION 6.22. *PROPORTIONALITY OF FOUR SEGMENTS TO THE POLYGONS CONSTRUCTED UPON THEM. If four segments are proportional, then if any pair of similar polygons are similarly constructed on the first and second segments, and if any other pair of similar polygons are constructed on the third and fourth segments, then these figures are proportional.*

Conversely, if we have that a polygon constructed on the first of four segments is similar and similarly constructed to the polygon constructed on the second segment as a polygon constructed on the third segment is similar and similarly constructed to the polygon constructed on the fourth segment, then the four lines are proportional.

PROOF. Construct four proportional segments (AB, CD, EF, GH) and similar polygons ($\triangle ABK$, $\triangle CDL$) constructed from AB and CD. Also construct similar polygons ($\boxdot MEFI$, $\boxdot NGHJ$) from the third and fourth segments (EI, GJ).

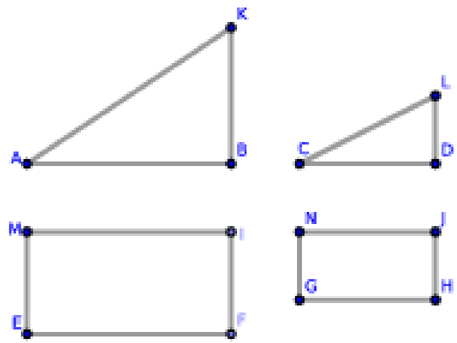

FIGURE 6.2.36. [6.22]

We prove each claim separately:

1. We claim that $\triangle ABK : \triangle CDL :: \boxdot HEFI : \boxdot NGHJ$.

Suppose that the segments AB, CD, EF, GH are proportional. Then we have that

$$\begin{array}{rcll} \triangle ABK : \triangle CDL & :: & AB^2 : CD^2 & [6.20] \\ \boxdot HEFI : \boxdot NGHJ & :: & EF^2 : GH^2 & [6.20] \\ AB : CD & :: & EF : GH & \text{(by hypothesis)} \end{array}$$

Since we have that $AB^2 : CD^2 :: EF^2 : GH^2$ by [5.22, Cor. 1], it follows that $\triangle ABK : \triangle CDL :: \boxdot HEFI : \boxdot NGHJ$.

2. Now suppose that $ABK : CDL :: \boxdot HEFI : \boxdot NGHJ$. We wish to show that $AB : CD :: EF : GH$.

Notice that

$$\begin{array}{rcll} ABK : CDL & :: & AB^2 : CD^2 & [6.20] \\ \boxdot HEFI : \boxdot NGHJ & :: & EF^2 : GH^2 & [6.20] \\ AB^2 : CD^2 & :: & EF^2 : GH^2 & [5.22, \text{Cor. } 1] \end{array}$$

Hence, $AB : CD :: EF : GH$. □

PROPOSITION 6.23. *EQUIANGULAR PARALLELOGRAMS. Equiangular parallelograms are to each other as the rectangles contained by their sides about a pair of equal angles.*

PROOF. Equiangular parallelograms ($\boxdot HABD$, $\boxdot BGEC$) are to each other as the rectangles contained by their sides about a pair of equal angles (or $X : Z :: AB.BD : BC.BG$).

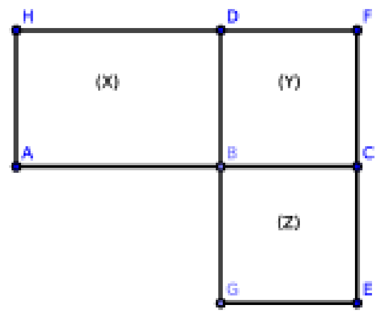

FIGURE 6.2.37. [6.23]

Let the two sides AB, BC about the equal angles $\angle ABD$, $\angle CBG$ be placed so as to form one segment AC; clearly, as in [6.14], GB, BD also form the segment GD. Complete the parallelogram $\boxdot DBCF$. Denoting the parallelograms $\boxdot HADB$, $\boxdot DBCF$, $\boxdot BGEC$ by X, Y, Z, respectively, we have that

$$\begin{array}{rcll} X : Y & :: & AB : BC & [6.1] \\ Y : Z & :: & BD : BG & [6.1] \end{array}$$

Hence, $XY : YZ :: AB.BD : BC.BG$, or $X : Z :: AB.BD : BC.BG$. □

Exercises.

1. Triangles which have one angle of one equal or supplemental to one angle of the other are to one another in the ratio of the rectangles of the sides about those angles.

2. Two quadrilaterals whose diagonals intersect at equal angles are to one another in the ratio of the rectangles of the diagonals.

PROPOSITION 6.24. *SIMILAR PARALLELOGRAMS ABOUT THE DIAGONAL. In any parallelogram, every two parallelograms which are about a diagonal are similar to the whole and to one another.*

PROOF. In any parallelogram ($\boxdot BADC$), every two parallelograms ($\boxdot EAGF$, $\boxdot JFHC$) which are about a diagonal are similar to the whole and to one another.

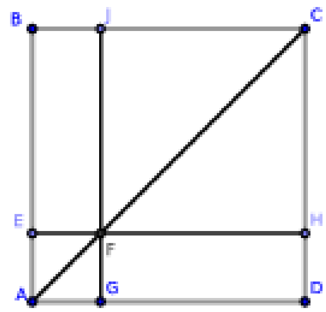

FIGURE 6.2.38. [6.24]

Since the parallelograms $\boxdot BADC$, $\boxdot EAGF$ have a common angle, they are equiangular [1.34], and all we are required to prove that that the sides about the equal angles are proportional.

Since the segments EF, are parallel, the triangles $\triangle AEF$, $\triangle ABC$ are equiangular; by [6.4], $AE : EF :: AB : BC$, and the other sides of the parallelograms are equal to AE, EF; AB, BC. Hence the sides about the equal angles are proportional, and therefore $\boxdot EAGF \sim \boxdot BADC$. The parallelograms $\boxdot EAGF$, $\boxdot JFHC$ may also be shown to be similar in the same way. $\square$

COROLLARY. *1. The parallelograms $\boxdot EAGF$, $\boxdot JFHC$, $\boxdot BADC$ are, respectively homothetic.*

PROPOSITION 6.25. *CONSTRUCTION OF A POLYGON EQUAL IN AREA TO A GIVEN FIGURE AND SIMILAR TO A SECOND GIVEN FIGURE.*

PROOF. We wish to construct a polygon equal to a given polygon, $ALMN$, and similar to a second polygon, BCD.

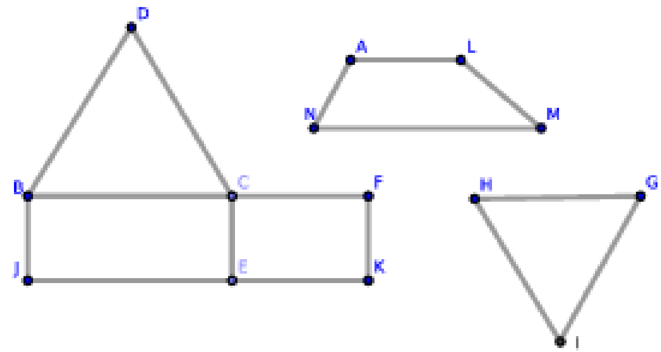

FIGURE 6.2.39. [6.25] Note that BCD and GHI need not be triangles.

On any side of the polygon BCD (wlog, we choose BC), construct the rectangle $\boxdot BJEC = BCD$ [1.45], and on CE construct the rectangle $\boxdot CEKF = ALMN$. Between BC, CF, find a mean proportional GH and on it construct the polygon $GHI \sim BCD$ [6.18] so that BC and GH may be homologous sides. We claim that GHI is the required polygon.

We have that $BC : GF :: GF : CF$; therefore $BC : CF$ is in the duplicate ratio of $BC : GH$ [Def 5.10]. Since $BCD \sim GHI$, $BCD : GHI$ is in the duplicate ratio of $BC : GH$ [6.20]. We also have that $BC : CF :: \boxdot BJEC : \boxdot CEKF$. Hence, $\boxdot BJCE : \boxdot CEKF :: BCD : GHI$. But the rectangle $\boxdot BJEC$ is equal to the polygon BCD; therefore, $\boxdot CEKF = GHI$. Recall that by construction $\boxdot CEKF = ALMN$. It follows that $GHI = ALMN$ and is similar to BCD. Hence, it is the required polygon. □

Alternatively:

PROOF. Construct the squares $\boxdot EFJK$, $\boxdot LMNO$ equal in area to the polygons BCD and $ALMN$, respectively [2.14]. Find GH, a fourth proportional to EF, LM, and BC [6.12].

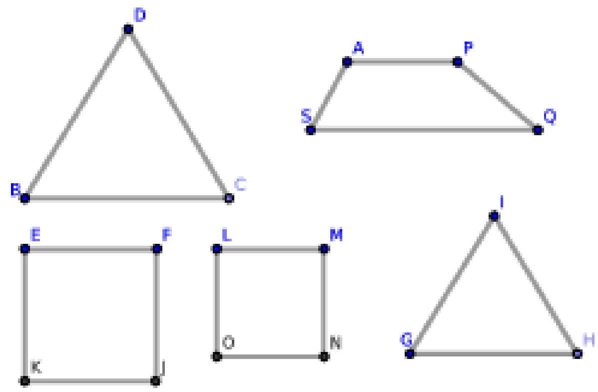

FIGURE 6.2.40. [6.25], alternate proof. Note that GHI need not be a triangle.

On GH, construct the polygon GHI similar to the polygon BCD [6.18] such that BC and GH are homologous sides. We claim that GHI is the required polygon.

Because $EF : LM :: BC : GH$ by construction, we have that $EFJK : LMNO :: BCD : GHI$ [6.22]. But $EFJK = BCD$ by construction; therefore, $LMNO = GHI$. But we also have that $LMNO = APQS$ by construction. Therefore $GHI = APQS$ and is similar to BCD. $\square$

PROPOSITION 6.26. *PARALLELOGRAMS ON A COMMON ANGLE. If two similar and similarly situated parallelograms have a common angle, they stand on the same diagonal.*

PROOF. If two similar and similarly situated parallelograms ($\boxdot AEFG$, $\boxdot ABCD$) have a common angle ($\angle GAF$), we claim that they stand on the same diagonal.

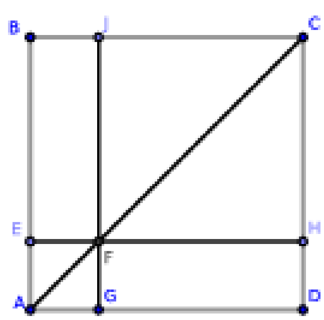

FIGURE 6.2.41. [6.26]

Construct the diagonals AF, AC. Because s $\boxdot AEFG \sim \boxdot ABCD$, they can be divided into the same number of similar triangles [6.20]. Hence, $\triangle GAF \sim \triangle CAD$, and it follows that $\angle GAF = \angle CAD$. Hence, AC must pass through the point F and hence the parallelograms are about the same diagonal. $\square$

Observation: [6.26] is the converse of [6.24] and may have been misplaced in an early edition of Euclid. The following would be a simpler statement of result: "If two homothetic parallelograms have a common angle, they are about the same diagonal."

PROPOSITION 6.27. *INSCRIBING A PARALLELOGRAM IN A TRIANGLE I.*

PROOF. Construct $\triangle ABC$. Bisect the side AC at point P opposite to the angle at point B. Through P, construct PE, PF parallel to the remaining sides of the triangle $\triangle ABC$. We claim that $\boxdot EBFP$ is the required parallelogram.

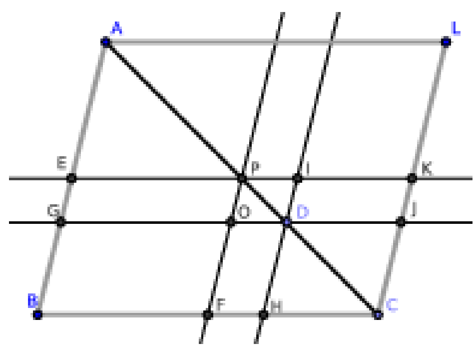

FIGURE 6.2.42. [6.27]

Take any other point D on AC, construct DG, DH parallel to the sides of $\triangle ABC$, and construct $CK \parallel AB$. Extend EP, GD to meet CK at K and J, and extend HD to meet PK at I.

Since AC is bisected at P, EK is also bisected in P. By [1.36], $\boxdot EGOP = \boxdot POJK$. Therefore, $\boxdot EGOP > \boxdot IDJK$; but $\boxdot IDJK = \boxdot OFHD$ [1.43], and so $\boxdot EGOP > \boxdot OFHD$. To each add $\boxdot GBFO$, and we have that $\boxdot EBFP > \boxdot GBHD$. Hence, $\boxdot EBFP$ is the maximum parallelogram which can be inscribed in the triangle $\triangle ABC$. $\square$

COROLLARY. *1. The maximum parallelogram exceeds any other parallelogram about the same angle in the triangle by the area of the similar parallelogram whose diagonal is the line between the midpoint P of the opposite side and the point D, which is the corner of the other inscribed parallelogram.*

COROLLARY. *2. The parallelograms inscribed in a triangle and having one angle common with it are proportional to the rectangles contained by the segments of the sides of the triangle made by the opposite corners of the parallelograms.*

COROLLARY. *3. The parallelogram in [6.27] has proportions $AC : GH :: AC^2 : AD.DC$.*

PROPOSITION 6.28. *INSCRIBING A PARALLELOGRAM IN A TRIANGLE II. We wish to inscribe in a given triangle a parallelogram equal to a given polygon not greater than the maximum inscribed parallelogram and having an angle common with the triangle.*

PROOF. We wish to inscribe a parallelogram in a given triangle ($\triangle ABC$) equal in area to a given polygon (XYZ) not greater than the maximum inscribed parallelogram and having an angle (at B) in common with the triangle.

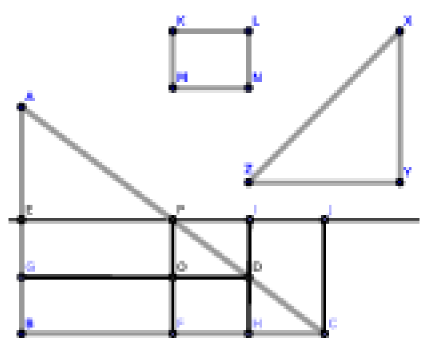

FIGURE 6.2.43. [6.28] Note that XYZ need not be a triangle.

Bisect the side AC at P, opposite to point B. Construct PF, PE parallel to the sides AB, BC. Then $\boxdot EBFP$ is the maximum parallelogram that can be inscribed in the triangle ABC [6.27]. If $XYZ = \boxdot EBFP$, the problem is solved.

Otherwise, extend EP and construct CJ parallel to PF; then construct the parallelogram $\boxdot KLMN$ equal to the difference between the areas between the polygons $\boxdot PJCF$ and XYZ [6.25] and similar to $\boxdot PJCF$ such that the sides PJ and KL will be homologous. Cut off $PI = KL$. Construct $IH \parallel AB$, cutting AC at D, and construct DG parallel to BC. We claim that $\boxdot GBHD$ is the required parallelogram.

Since the parallelograms $\boxdot PFCJ$, $\boxdot PODI$ stand on the same diagonal, they are similar [6.24]; however, $\boxdot PFCJ \sim \boxdot KLMN$ by construction, and therefore $\boxdot PODI \sim \boxdot KLMN$. Also by construction, their homologous sides, PI and KL, are equal. Hence by [6.20], $\boxdot PODI = \boxdot KLMN$. Now, $\boxdot PODI$ is the difference between $\boxdot EBFP$ and $\boxdot GBHD$ [6.27, Cor. 1], and $\boxdot KLMN$ is the difference between $\boxdot PJCF$ and XYZ by construction. Therefore, the difference between $\boxdot PJCF$ and XYZ is equal to the difference between $\boxdot EBFP$ and $\boxdot GBHD$. However, $\boxdot EBFP = \boxdot PJCF$. Hence, $\boxdot GBHD = XYZ$. $\square$

PROPOSITION 6.29. *ESCRIBING A PARALLELOGRAM TO A TRIANGLE. We wish to escribe to a given triangle a parallelogram equal to a given polygon and having an angle common with an external angle of the triangle.*

PROOF. We wish to escribe to a given triangle ($\triangle ABC$) a parallelogram equal in area to a given polygon (XYZ) and having an angle common with an external angle (at B) of the triangle.

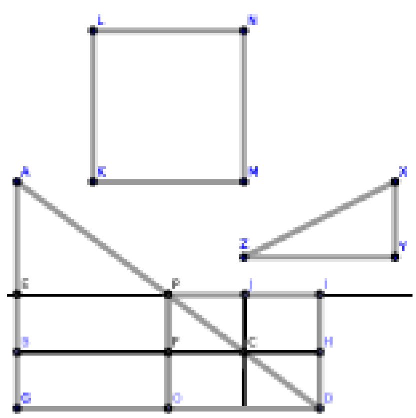

FIGURE 6.2.44. [6.29]

The construction is the same as [6.28] except that we construct the parallelogram $\boxdot KLMN$ so that its area is equal to the sum of parallelograms

$\boxdot PODI$ and $\boxdot XYZ$. Make $PI = KL$ and construct $IH \parallel AB$; the remaining construction takes place as in [6.28].

Now as in [2.6], it can be shown that the parallelogram $\boxdot BGDH$ is equal in area to the gnomon OHJ: that is, the area of $\boxdot BGDH$ equals to the difference between the parallelograms $\boxdot PODI$ and $\boxdot PFCJ$. By construction, this is also the difference between $\boxdot KLNM$ and $\boxdot PFCJ$, which is equal in area of XYZ. Notice that $\boxdot BGDH$ is escribed to the triangle $\triangle ABC$ and has an angle common with the external angle at B. Hence, the proof. □

PROPOSITION 6.30. *We wish to divide a given segment into its "extreme and mean ratio."*

PROOF. We wish to divide a given segment (AB) into its "extreme and mean ratio."

FIGURE 6.2.45. [6.30]

Divide AB at C so that the rectangle $AB.BC = AC^2$ [2.11]. We claim that C is the required point.

Because the rectangle $AB.BC = AC^2$, we have that $AB : AC :: AC : BC$ [6.17]. Hence AB is cut in extreme and mean ratio at C [Def. 6.2]. □

Exercises.

1. If the three sides of a right triangle are in continued proportion, the hypotenuse is divided in extreme and mean ratio by the perpendicular from the right angle on the hypotenuse.

2. In the same case as #1, the greater segment of the hypotenuse is equal to the least side of the triangle.

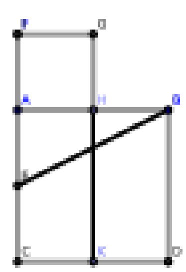

FIGURE 6.2.46. [6.30], #3

3. The square on the diameter of the circle constructed about the triangle formed by the points F, H, D (see Fig. 6.2.46) is equal to six times the square on the segment FD.

PROPOSITION 6.31. *AREA OF SQUARES ON A RIGHT TRIANGLE. If any similar polygon is similarly constructed on the three sides of a right triangle, the polygon on the hypotenuse is equal in area to the sum of the areas of those polygons constructed on the two other sides.*

PROOF. If any similar polygon is similarly constructed on the three sides of a right triangle ($\triangle ABC$), we claim that the polygon on the hypotenuse is equal in area to the sum of the areas of those polygons constructed on the two other sides.

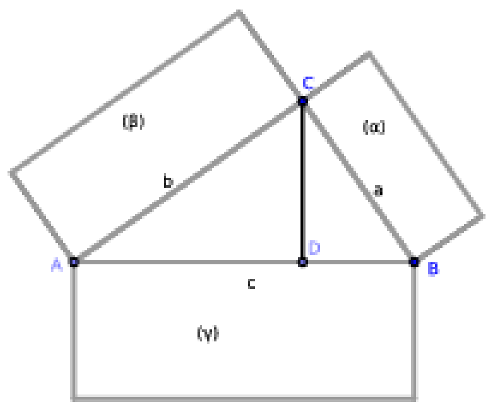

FIGURE 6.2.47. [6.31]

Construct $CD \perp AB$ [1.12]. Because $\triangle ABC$ is a right triangle and CD is constructed from the right angle perpendicular to the hypotenuse, $BD : AD$ is in the duplicate ratio of $BA : AC$ [6.8, Cor. 4]. Again, because the polygons constructed on BA, AC are similar, they are in the duplicate ratio of $BA : AD$ [6.20]. Hence by [5.11],

$BA : AD$ :: figure constructed on BA : figure constructed on AC

Similarly,

$AB : BD$ :: figure constructed on AB : figure constructed on BC

By [5.24],

$AB : AD + BD$ :: figure constructed on AB : sum of the figures on AC, BC

But $AB = AD+BD$. Therefore the polygon constructed on the segment AB is equal in area to the sum of the similar polygons constructed on the segments AC and BC. □

Alternatively:

PROOF. Denote the sides by a, b, c, and the polygons by α, β, γ. Because the polygons are similar, by [6.20] we have that $\alpha : \gamma :: a^2 : c^2$.

Therefore,

$$\frac{\alpha}{\gamma} = \frac{a^2}{c^2}$$

Similarly,

$$\frac{\beta}{\gamma} = \frac{b^2}{c^2}$$

from which it follows that

$$\frac{\alpha+\beta}{\gamma} = \frac{a^2+b^2}{c^2}$$

But $a^2 + b^2 = c^2$ by [1.47]. Therefore, $\alpha + \beta = \gamma$; or, the sum of the polygons on the sides is equal to the polygon on the hypotenuse. □

Exercise.

1. If semicircles are constructed on supplemental chords of a semicircle, the sum of the areas of the two crescents thus formed is equal to the area of the triangle whose sides are the supplemental chords and the diameter.

PROPOSITION 6.32. *FORMATION OF TRIANGLES. If two triangles which have two sides of one triangle proportional to two sides of the other triangle and the contained equal angles are joined at an angle so as to have their homologous sides parallel, the remaining sides are in the same segment.*

PROOF. If two triangles ($\triangle ABC$, $\triangle CDE$) which have two sides of one triangle proportional to two sides of the other triangle ($AB : BC :: CD : DE$) and the contained equal angles (at points B, D) are joined at an angle (at C) so as to have their homologous sides parallel, the remaining sides (AC, CE) are in the same segment (AE).

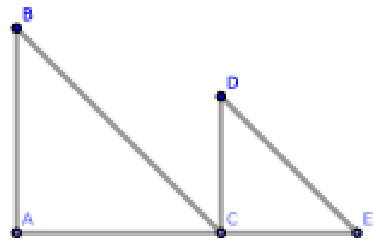

FIGURE 6.2.48. [6.32]

Because the triangles $\triangle ABC$, $\triangle CDE$ have equal angles at B and D and the sides about these angles are proportional, ($AB : BC :: CD : DE$), they are equiangular [6.6]. Therefore, $\angle BAC = \angle DCE$. To each add $\angle ACD$, and we have $\angle BAC + \angle ACD = \angle DCE + \angle ACD$. But $\angle BAC + \angle ACD$ is two right angles [1.29]. It follows that $\angle DCE + \angle ACD$ is two right angles. Hence by [1.47], AC, CE are in the same segment. □

PROPOSITION 6.33. *RATIOS OF EQUAL TRIANGLES. In equal circles, angles at the centers or at the circumferences have the same ratio to one another as the arcs on which they stand. This also holds true for sectors.*

PROOF. In equal circles ($\circ ABC$, $\circ DEF$), angles at the centers ($\angle BOC$, $\angle EPF$) or at the circumferences ($\angle BAC$, $\angle EDF$) have the same ratio to one another as the arcs (BC, EF) on which they stand. This also holds true for sectors (BOC, EPF).

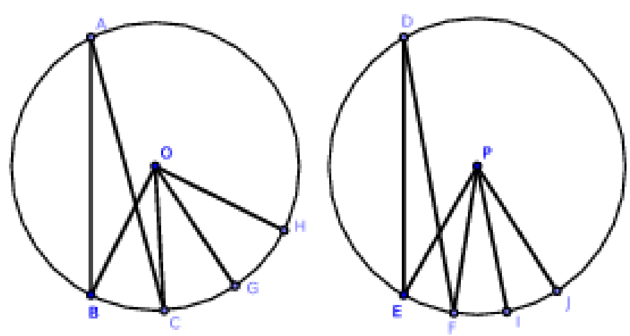

FIGURE 6.2.49. [6.33]

We prove each claim separately:

1. We claim that $BC : EF :: \angle BAC : \angle EDF$.

Take any number of arcs CG, GH in the first circle, each equal in length to BC. Join OG, OH. In the second circle, take any number of arcs FI, IJ, each equal to EF. Join IP, JP. Because the arcs $BC = CG = GH$, we have that $\angle BOC = \angle COG = \angle GOH$ [3.27]. Therefore, the arc BH and $\angle BOH$ are equimultiples of the arc BC and $\angle BOC$.

Similarly, it may be shown that the arc EJ and $\angle EPJ$ are equimultiples of the arc EF and $\angle EPF$. Again, since the circles are equal, it is evident that $\angle BOH$ is greater than, equal to, or less than $\angle EPJ$ according to whether the arc BH is greater than, equal to, or less than the arc EJ. We have four magnitudes: the arc BC, the arc EF, the angle $\angle BOC$, and the angle $\angle EPF$. We have also taken equimultiples of the first and third (the arc BH and $\angle BOH$) and other equimultiples of the second and fourth (the arc EJ and $\angle EPJ$) and we have proved that, according as the multiple of the first is greater than, equal to, or less than the multiple of the second, the multiple of the third is greater than, equal to, or less than the multiple of the fourth.

Hence by [Def. 5.5], $BC : EF :: \angle BOC : \angle EPF$. Again, since the angle $\angle BAC$ is half the angle $\angle BOC$ [3.20.] and $\angle EDF$ is half the angle $\angle EPF$, we have that

$$\angle BOC : \angle EPF \quad :: \quad \angle BAC : \angle EDF \quad [5.15]$$

from which it follows that $BC : EF :: \angle BAC : \angle EDF$ [5.11].

2. We claim that sector BOC : sector $EPF :: BC : EF$.

We make the same construction as in part 1. Since the arc BC is equal in length to CG, $\angle BOC = \angle COG$. Hence the sectors BOC, COG are congruent (see Observation, [3.29]); therefore, they are equal in length. Similarly, the sectors COG, GOH are equal in length. Hence there are as many equal sectors as there are equal arcs; therefore, the arc BH and the sector BOH are equimultiples of the arc BC and the sector BOC. Similarly, it may be shown that the arc EJ and the sector EPJ are equimultiples of the arc EF and the sector EPF. And it is evident by superposition that if the arc BH is greater than, equal to, or less than the arc EJ, the sector BOH is greater than, equal to, or less than the sector EPJ. Hence by [Def. 5.5], arc BC : arc EF :: sector BOC : sector EPF. □

Alternative proof to part 2):

PROOF. Sector $BOC = (\frac{1}{2}$ rectangle by arc $BC)$ and the radius of the circle $\circ ABC$ [6.20, #14], and sector $EPF = (2\cdot$ rectangle contained by the arc EF and the radius of the circle $\circ EDF)$. Since the circles are equal, their radii are equal. Hence, sector BOC : sector EPF :: arc BC : arc EF. □

Examination questions for chapter 6.

1. What is the subject-matter of chapter 6? (Ans. Application of the theory of proportion.)

2. What are similar polygons?

3. What do similar polygons agree in?

4. How many conditions are necessary to define similar triangles?

5. How many conditions are necessary to define similar polygons of more than three sides?

6. When is a polygon said to be given in species?

7. When in magnitude?

8. When in position?

9. What is a mean proportional between two lines?

10. Define two mean proportionals.

11. What is the altitude of a polygon?

12. If two triangles have equal altitudes, how do their areas vary?

13. How do these areas vary if they have equal bases but unequal altitudes?

14. If both bases and altitudes differ, how do the areas vary?

15. When are two lines divided proportionally?

16. If in two segments which are divided proportionally a pair of homologous points coincide with their point of intersection, what property holds for the lines joining the other pairs of homologous points?

17. Define reciprocal proportion.

18. If two triangles have equal areas, prove that their perpendiculars are reciprocally proportional to the bases.

19. What is meant by inversely similar polygons?

20. If two polygons are inversely similar, how can they be changed into polygons which are directly similar?

21. Give an example of two triangles inversely similar. (Ans. If two lines passing through any point O outside a circle intersect it in pairs of points A, A'; B, B', respectively, the triangles $\triangle OAB$, $\triangle OA'B'$ are inversely similar.)

22. What point is it round which a polygon can be turned so as to bring its sides into positions of parallelism with the sides of a similar polygon? (Ans. The center of similitude of the two polygons.)

23. How many polygons similar to a given polygon of sides can be constructed on a given line?

24. How many centers of similitude can two regular polygons of n sides each have? (Ans. n centers, which lie on a circle.)

25. What are homothetic polygons?

26. How do the areas of similar polygons vary?
27. What proposition is [6.19] a special case of?
28. Define Philo's line.
29. How many centers of similitude do two circles have?

Exercises for chapter 6.

1. If in a fixed triangle, we construct a variable side parallel to the base, the locus of the points of intersection of the diagonals of the trapezium that is cut off from the triangle is the median that bisects the base.

2. Find the locus of the point which divides in a given ratio the several lines constructed from a given point to the circumference of a given circle.

3. Two segments AB, XY, are given in position: AB is divided at C in the ratio $m : n$ and parallels AA', BB', CC' are constructed in any direction meeting XY in the points A', B', C'. Prove that

$$(m+n)CC' = nAA' + mBB'$$

4. Three concurrent lines from the vertices of a triangle $\triangle ABC$ meet the opposite sides in A', B', C'. Prove $AB'.BC'.CA' = A'B.B'C.C'A$.

5. If a transversal meets the sides of a triangle $\triangle ABC$ at the points A', B', C', prove $AB'.BC'.CA' = -A'B.B'C.C'A$.

6. If on a variable segment AC which is constructed from a fixed point A to any point B on the circumference of a given circle, a point C is taken such that the rectangle $AB.AC$ is constant, prove that the locus of C is a circle.

7. If D is the midpoint of the base BC of a triangle $\triangle ABC$, E the foot of the perpendicular, L is the point where the bisector of the angle at A meets BC, and H the point of intersection of the inscribed circle with BC, prove that $DE.HL = HE.HD$.

8. As in #7, if K is the point of intersection with BC of the escribed circle, which touches the other extended sides, prove that $LH.BK = BD.LE$.

9. If R, r, r', r'', r''' are the radii of the circumscribed, the inscribed, and the escribed circles of a plane triangle, d, d', d'', d''' the distances of the center of the circumscribed circle from the centers of the others, then $R^2 = d^2 + 2Rr = d'^2 - 2Rr'$, etc.

10. As in #9, prove that $12R^2 = d^2 + d'^2 + d''^2 + d'''^2$.

11. If p', p'', p''' denote the altitudes of a triangle, then:

(1)	$\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}$	$=$	$\frac{1}{r}$	
(2)	$\frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p}$	$=$	$\frac{1}{r'}$	(etc.)
(3)	$\frac{2}{p}$	$=$	$\frac{1}{r} - \frac{1}{r'}$	(etc.)
(4)	$\frac{2}{p'}$	$=$	$\frac{1}{r''} + \frac{1}{r'''}$	(etc.)

12. In a given triangle, inscribe another of given form which has one of its angles at a given point in one of the sides of the original triangle.

13. If a triangle of given form moves so that its three sides pass through three fixed points, the locus of any point in its plane is a circle.

14. Suppose that the angle at point A and the area of a triangle $\triangle ABC$ are given in magnitude. If the point A is fixed in position and the point B move along a fixed line or circle, then the locus of the point C is a circle.

15. One of the vertices of a triangle of given form remains fixed, and the locus of another is a segment or circle. Find the locus of the third.

16. Find the area of a triangle:

(a) in terms of its medians;

(b) in terms of its perpendiculars.

17. If two circles touch externally, their common tangent is a mean proportional between their diameters.

18. If there are three given parallel lines and two fixed points A, B, and if the lines connecting A and B to any variable point in one of the parallels intersects the other parallels at the points C and D, E and F, respectively, prove that CF and DE each pass through a fixed point.

19. If a system of circles pass through two fixed points, any two secants passing through one of the points are cut proportionally by the circles.

20. Find a point O in the plane of a triangle $\triangle ABC$ such that the diameters of the three circles about the triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCA$ may be in the ratios of three given segments.

21. Suppose that $ABCD$ is a cyclic quadrilateral, and the segments AB, AD, and the point C are given in position. Find the locus of the point which divides BD in a given ratio.

22. If CA, CB are two tangents to a circle and $BE \perp AD$ (where AD is the the diameter through A), then prove that CD bisects BE.

23. If three segments from the vertices of a triangle $\triangle ABC$ to any interior point O meet the opposite sides in the points A', B', C', prove that

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1$$

24. If three concurrent lines OA, OB, OC are cut by two transversals in the two systems of points A, B, C; A', B', C', respectively, then prove that

$$\frac{AB}{A'B'} \cdot \frac{OC}{OC'} = \frac{BC}{B'C'} \cdot \frac{OA}{OA'} = \frac{CA}{C'A'} \cdot \frac{OB}{OB'}$$

25. The line joining the midpoints of the diagonals of a quadrilateral circumscribed to a circle:

(a) divides each pair of opposite sides into inversely proportional segments;

(b) is divided by each pair of opposite segments into segments which when measured from the center are proportional to the sides;

(c) is divided by both pairs of opposite sides into segments which when measured from either diagonal have the same ratio to each other.

26. If CD, CD' are the internal and external bisectors of the angle at C of the triangle $\triangle ACB$, the three rectangles $AD.DB$, $AC.CB$, $AD.BD$ are proportional to the squares of AD, AC, AD and are:

(a) in arithmetical progression, if the difference of the base angles is equal to a right angle;

(b) in geometrical progression if one base angle is right;

(c) in harmonic progression if the sum of the base angles is equal to a right angle.

27. If a variable circle touches two fixed circles, the chord of contact passes through a fixed point on the line connecting the centers of the fixed circles.

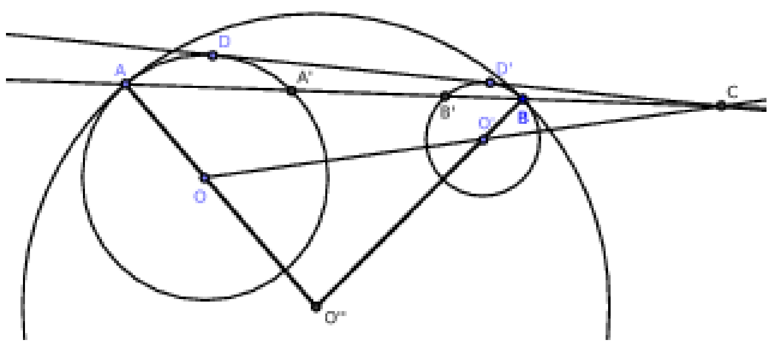

FIGURE 6.2.50. Ch. 6, #27

Let O,O' be the centers of the two fixed circles; O the center of the variable circle; A, B the points of contact. Let AB and OO' meet at C, and cut the fixed circles again in the points A', B' respectively. Join $A'O$, AO, BO'. Then AO, BO' meet at O'' [3.11]. Now because the triangles OAA', $O''AB$ are isosceles, the angles $O''BA = O''AB = OA'A$. Hence $OA' \parallel O'B$; therefore $OC : O'C :: OA' : O'B$ is in a given ratio. Hence, C is a given point.

28. If DD' is the common tangent to the two circles, then $DD'^2 = AB'.A'B$.

29. If R denotes the radius of O'' and ρ, ρ' the radii of O, O', then $DD'^2 : AB^2 :: (R \pm \rho)(R \pm \rho') : R^2$ where the choice of sign depends on the nature of the contacts. (This result follows from #28.)

30. If four circles are tangential to a fifth, and if we denote by $\overline{12}$ the common tangent to the first and second, etc., then $\overline{12}.\overline{34} + \overline{23}.\overline{14} = \overline{13}.\overline{24}$.

31. The inscribed and escribed circles of any triangle are all touched by its nine-points circle.

32. The four triangles which are determined by four points, taken three by three, are such that their nine-points circles have one common point.

33. If a, b, c, d denote the four sides, and D, D' the diagonals of a quadrilateral, prove that the sides of the triangle, formed by joining the feet of the perpendiculars from any of its angular points on the sides of the triangle formed by the three remaining points, are proportional to the three rectangles ac, bd, DD'.

34. Prove the converse of Ptolemy's theorem (see [6.17], #13).

35. Construct a circle which:

(a) passes through a given point, and touch two given circles;

(b) touches three given circles.

36. If a variable circle touches two fixed circles, the tangent to it from their center of similitude through which the chord of contact passes is of constant length (see Fig. 6.2.50).

37. If the lines AD, BD' are extended (see Fig. 6.2.50), they meet at a point on the circumference of O'', and the line $O''P$ is perpendicular to DD'.

38. If A, B are two fixed points on two lines given in position, and A', B' are two variable points such that the ratio $AA' : BB'$ is constant, the locus of the point dividing $A'B'$ in a given ratio is a segment.

39. If a segment EF divides proportionally two opposite sides of a quadrilateral, and a segment GH the other sides, each of these is divided by the other in the same ratio as the sides which determine them.

40. In a given circle, inscribe a triangle such that the triangle whose angular points are the feet of the perpendiculars from the endpoints of the base on the bisector of the vertical angle and the foot of the perpendicular from the vertical angle on the base may be a maximum.

41. In a circle, the point of intersection of the diagonals of any inscribed quadrilateral coincides with the point of intersection of the diagonals of the circumscribed quadrilateral whose sides touch the circle at the angular points of the inscribed quadrilateral.

42. Through two given points describe a circle whose common chord with another given circle may be parallel to a given line, or pass through a given point.

43. Being given the center of a circle, describe it so as to cut the legs of a given angle along a chord parallel to a given line.

44. If concurrent lines constructed from the angles of a polygon of an odd number of sides divide the opposite sides each into two segments, the product of one set of alternate segments is equal in area to the product of the other set.

45. If a triangle is constructed about a circle, the lines from the points of contact of its sides with the circle to the opposite angular points are concurrent.

46. If a triangle is inscribed in a circle, the tangents to the circle at its three angular points meet the three opposite sides at three collinear points.

47. The external bisectors of the angles of a triangle meet the opposite sides in three collinear points.

48. Construct a circle touching a given line at a given point and cutting a given circle at a given angle.

Definition: the center of mean position of any number of points A, B, C, D, etc., is a point which may be found as follows: bisect the line joining any two points A, B at G. Join G to a third point C; divide GC at H so that $GH = \frac{1}{3}GC$. Join H to a fourth point D and divide HD at K, so that $HK = \frac{1}{4}HD$, and so on. The last point found will be the center of mean position of the given points.

49. The center of mean position of the angular points of a regular polygon is the center of figure of the polygon.

50. The sum of the perpendiculars let fall from any system of points A, B, C, D, etc., whose number is n on any line L, is equal to n times the perpendicular from the center of mean position on L.

51. The sum of the squares of segments constructed from any system of points A, B, C, D, etc., to any point P exceeds the sum of the squares of segments from the same points to their center of mean position, O, by nOP^2.

52. If a point is taken within a triangle so as to be the center of mean position of the feet of the perpendiculars constructed from it to the sides of the triangle, the sum of the squares of the perpendiculars is a minimum.

53. Construct a quadrilateral being given two opposite angles, the diagonals, and the angle between the diagonals.

54. A circle rolls inside another of double its diameter; find the locus of a fixed point in its circumference.

55. Two points, C, D in the circumference of a given circle are on the same side of a given diameter. Find a point P in the circumference at the other side of the given diameter, AB, such that PC, PD may cut AB at equal distances from the center.

56. If the sides of any polygon be cut by a transversal, the product of one set of alternate segments is equal to the product of the remaining set.

57. A transversal being constructed cutting the sides of a triangle, the lines from the angles of the triangle to the midpoints of the segments of the transversal intercepted by those angles meet the opposite sides in collinear points.

58. If segments are constructed from any point P to the angles of a triangle, the perpendiculars at P to these segments meet the opposite sides of the triangle at three collinear points.

59. Divide a given semicircle into two parts by a perpendicular to the diameter so that the radii of the circles inscribed ob them may have a given ratio.

60. From a point within a triangle, suppose that perpendiculars fall on the sides; find the locus of the point when the sum of the squares of the lines joining the feet of the perpendiculars is given.

61. If a circle makes given intercepts on two fixed lines, the rectangle contained by the perpendiculars from its center on the bisectors of the angle formed by the lines is given.

62. If the base and the difference of the base angles of a triangle are given, the rectangle contained by the perpendiculars from the vertex on two lines through the midpoint of the base, parallel to the internal and external bisectors of the vertical angle, is constant.

63. The rectangle contained by the perpendiculars from the endpoints of the base of a triangle on the internal bisector of the vertical angle is equal to the rectangle contained by the external bisector and the perpendicular from the middle of the base on the internal bisector.

64. State and prove the corresponding theorem for perpendiculars on the external bisector.

65. Suppose that R, R' denote the radii of the circles inscribed in the triangles into which a right triangle is divided by the perpendicular from the right angle on the hypotenuse. If c is the hypotenuse and s is the semi-perimeter, $R^2 + R'^2 = (s-c)^2$.

66. If A, B, C, D are four collinear points, find a point O in the same line with them such that $OA.OD = OB.O$C.

67. Suppose the four sides of a cyclic quadrilateral are given; construct it.

68. Being given two circles, find the locus of a point such that tangents from it to the circles may have a given ratio.

69. If four points A, B, C, D are collinear, find the locus of the point P at which AB and CD stand opposite equal angles.

70. If a circle touches internally two sides of a triangle, CA, CB, and its circumscribed circle, the distance from C to the point of intersection on either side is a fourth proportional to the semi-perimeter, CA, and CB.

71. State and prove the corresponding theorem for a circle touching the circumscribed circle externally and two extended sides.

72. Pascal's Theorem: if the opposite sides of an irregular hexagon $ABCDEF$ inscribed in a circle are extended until they meet, the three points of intersection G, H, I are collinear. See Fig. 6.2.51.

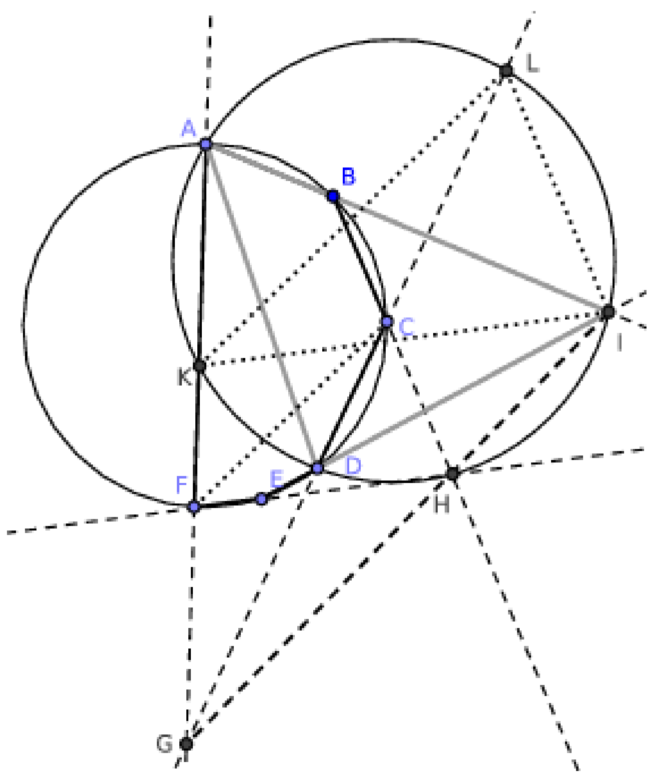

FIGURE 6.2.51. Ch. 6, #72. Pascal's Theorem

Join AD. Construct a circle about the triangle $\triangle ADI$, cutting the extended segments AF, CD, if necessary, at K and L. Join IK, KL, LI. By [3.21], we have that $\angle KLG = \angle FCG = \angle GAD$. Therefore $KL \parallel CF$. Similarly, $LI \parallel CH$ and $KI \parallel FH$; hence the triangles $\triangle KLI$, $\triangle FCH$ are homothetic, and so the lines joining corresponding vertices are concurrent. Therefore, the points I, H, G are collinear.

73. If two sides of a triangle circumscribed to a given circle are given in position with the third side variable, the circle constructed about the triangle touches a fixed circle.

74. If two sides of a triangle are given in position, and if the area is given in magnitude, two points can be found at each of which the base stands opposite a constant angle.

75. If a, b, c, d denote the sides of a cyclic quadrilateral and s its semi-perimeter, prove that its area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

76. If three concurrent lines from the angles of a triangle $\triangle ABC$ meet the opposite side in the points A', B', C', and the points A', B', C' are joined, forming a second triangle $\triangle A'B'C'$, then

$$\triangle ABC : \triangle A'B'C' :: AB.BC.CA : 2AB'.BC'.CA'$$

77. In the same case as #76, find the diameter of the circle circumscribed about the triangle $\triangle ABC = AB'.BC'.CA'$ divided by the area of $A'B'C'$.

78. If a quadrilateral is inscribed in one circle and circumscribed to another, the square of its area is equal to the product of its four sides.

79. If on the sides AB, AC of a triangle $\triangle ABC$ we take two points D, E on their connecting segment such that

$$\frac{BD}{AD} = \frac{AE}{CE} = \frac{DE}{EF}$$

then prove that the triangle $BFC = 2ADE$.

80. If through the midpoints of each of the two diagonals of a quadrilateral we construct a parallel to the other, the lines constructed from their points of intersection to the midpoints of the sides divide the quadrilateral into four equal parts.

81. Suppose that CE, DF are perpendiculars to the diameter of a semicircle, and two circles are constructed touching CE, DE, and the semicircle, one internally and the other externally. Prove that the rectangle contained by the perpendiculars from their centers on AB is equal to $CE.DF$.

82. If segments are constructed from any point in the circumference of a circle to the angular points of any inscribed regular polygon of an odd number of sides, the sums of the alternate lines are equal.

83. If at the endpoints of a chord constructed through a given point within a given circle tangents are constructed, the sum of the reciprocals of the perpendiculars from the point upon the tangents is constant.

84. If a cyclic quadrilateral is such that three of its sides pass through three fixed collinear points, the fourth side passes through a fourth fixed point, collinear with the three given ones.

85. If all the sides of a polygon are parallel to given lines and if the loci of all the angles except for one are segments, the locus of the remaining angle is also a segment.

86. If the vertical angle and the bisector of the vertical angle is given, the sum of the reciprocals of the containing sides is constant.

87. If P, P' denote the areas of two regular polygons of any common number of sides inscribed and circumscribed to a circle, and Π, Π' are the areas of the corresponding polygons of double the number of sides, prove that Π is a geometric mean between P and P' and Π' a harmonic mean between Π and P.

88. The difference of the areas of the triangles formed by joining the centers of the circles constructed about the equilateral triangles constructed outwards on the sides of any triangle is equal to the area of that triangle. Prove the same if they are constructed inwards.

89. In the same case as #88, the sum of the squares of the sides of the two new triangles is equal to the sum of the squares of the sides of the original triangle.

90. Suppose that R, r denote the radii of the circumscribed and inscribed circles to a regular polygon of any number of sides, R', r', corresponding radii to a regular polygon of the same area, and double the number of sides. Prove that $R' = \sqrt{Rr}$ and $r' = \sqrt{\frac{r(R+r)}{2}}$.

91. If the altitude of a triangle is equal to its base, the sum of the distances of the orthocenter from the base and from the midpoint of the base is equal to half the base.

92. In any triangle, when the base and the ratio of the sides are given, the radius of the circumscribed circle is to the radius of the circle which is the locus of the vertex as the difference of the squares of the sides is to four times the area.

93. Given the area of a parallelogram, one of its angles, and the difference between its diagonals, construct the parallelogram.

94. If a variable circle touches two equal circles, one internally and the other externally, and perpendiculars fall from its center on the transverse tangents to these circles, the rectangle of the intercepts between the feet of these perpendiculars and the intersection of the tangents is constant.

95. Given the base of a triangle, the vertical angle, and the point in the base whose distance from the vertex is equal half the sum of the sides, construct the triangle.

96. If the midpoint of the base BC of an isosceles triangle $\triangle ABC$ is the center of a circle touching the equal sides, prove that any variable tangent to the circle will cut the sides in points D, E, such that the rectangle $BD.CE$ is constant.

97. Inscribe in a given circle a trapezium, the sum of whose opposite parallel sides is given and whose area is given.

98. Inscribe in a given circle a polygon all of whose sides pass through given points.

99. If two circles $\circ ABC$, $\circ XYZ$ are related such that a triangle may be inscribed in $\circ ABC$ and circumscribed about $\circ XYZ$, prove that an infinite number of such triangles can be constructed.

100. In the same case as #99: the circle inscribed in the triangle formed by joining the points of contact on $\circ XYZ$ touches a given circle.

101. In the same case as #99: the circle constructed about the triangle formed by drawing tangents to $\circ ABC$ at the angular points of the inscribed triangle touches a given circle.

102. Find a point, the sum of whose distances from three given points is a minimum.

103. A line constructed through the intersection of two tangents to a circle is divided harmonically by the circle and the chord of contact.

104. Construct a quadrilateral similar to a given quadrilateral whose four sides pass through four given points.

105. Construct a quadrilateral similar to a given quadrilateral whose four vertices lie on four given lines.

106. Given the base of a triangle, the difference of the base angles, and the rectangle of the sides, construct the triangle.

107. Suppose that $\boxdot ABCD$ is a square, the side CD is bisected at E, and the line EF is constructed making the angle $\angle AEF = \angle EAB$. Prove that EF divides the side BC in the ratio of $2 : 1$.

108. If any chord is constructed through a fixed point on a diameter of a circle, its endpoints are joined to either end of the diameter, and the joining lines cut off on the tangent at the other end, then the portions whose rectangle is constant.

109. If two circles touch and through their point of intersection two secants be constructed at right angles to each other, cutting the circles respectively in the points A, A'; B, B'; then $AA'^2 + BB'^2$ is constant.

110. If two secants stand at right angles to each other which pass through one of the points of intersection of two circles also cut the circles again, and the line through their centers is the two systems of points a, b, c; a', b', c' respectively, then $ab : bc :: a'b' : b'c'$.

111. If a chord of a given circle stands opposite a right angle at a given point, the locus of the intersection of the tangents at its endpoints is a circle.

112. The rectangle contained by the segments of the base of a triangle made by the point of intersection of the inscribed circle is equal to the rectangle

contained by the perpendiculars from the endpoints of the base on the bisector of the vertical angle.

113. If O is the center of the inscribed circle of the triangle, prove

$$\frac{OA^2}{bc} + \frac{OB^2}{ca} + \frac{OC^2}{ab} = 1$$

114. State and prove the corresponding theorems for the centers of the escribed circles.

115. Suppose that four points A, B, C, D are collinear. Find a point P at which the segments AB, BC, CD stand opposite equal angles.

116. The product of the bisectors of the three angles of a triangle whose sides are a, b, c, is

$$\frac{8abc.s.\text{area}}{(a+b)(b+c)(c+a)}$$

117. In the same case as #116, the product of the alternate segments of the sides made by the bisectors of the angles is

$$\frac{a^2b^2c^2}{(a+b)(b+c)(a+c)}$$

118. If three of the six points in which a circle meets the sides of any triangle are such that the lines joining them to the opposite vertices are concurrent, the same property is true of the three remaining points.

119. If a triangle $\triangle A'B'C'$ is inscribed in another $\triangle ABC$, prove

$$AB'.BC'.CA' + A'B.B'C.C'A$$

is equal to twice the triangle $\triangle A'B'C'$ multiplied by the diameter of the circle $\circ ABC$.

120. Construct a polygon of an odd number of sides being given that the sides taken in order are divided in given ratios by fixed points.

121. If the external diagonal of a quadrilateral inscribed in a given circle is a chord of another given circle, the locus of its midpoint is a circle.

122. If a chord of one circle is a tangent to another, the line connecting the midpoint of each arc which it cuts off on the first to its point of intersection with the second passes through a given point.

123. From a point P in the plane of a given polygon, suppose that perpendiculars fall on its sides. If the area of the polygon formed by joining the feet of the perpendiculars is given, the locus of P is a circle.

124. The medians of a triangle divide each other in the ratio of $2:1$.

CHAPTER 7

Infinite Primes

This brief chapter highlights Proposition IX.20, Euclid's proof of infinitely many prime numbers.

7.1. Definitions

1. The set of all positive integers (1, 2, 3, ...) is called the set of *natural numbers*.

2. A natural number that has only one pair of factors, namely 1 and itself, is a *prime number*.

3. A natural number that has more than one pair of factors is a *composite number*.

4. The only natural number that is neither prime nor composite is 1.

7.2. The Proposition

This proof is based on Chris K. Caldwell's[1] proof which is in turn based on Euclid's original[2].

PROPOSITION 7.1. *Any list of prime numbers which is finite is incomplete.*

PROOF. Suppose we obtain a list of all prime numbers, and the list is finite. Call the primes in our finite list p_1, p_2, ..., p_r. Let P be any common multiple of these primes plus one (for example, $P = p_1 \cdot p_2 \cdot ... \cdot p_r + 1$).

Now P is either prime or it is not. If it is prime, then P is a prime that was not in our list.

If P is not prime, then it is divisible by some prime: call it p. Notice p cannot be any of p_1, p_2, ..., p_r, otherwise p would divide 1, a contradiction since the natural numbers do not contain fractions. So this prime p is some prime that was not in our original list.

In either case, the original finite list was incomplete, and hence the actual list must be infinite in length. □

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[2] `http://primes.utm.edu/notes/proofs/infinite/euclids.html`

CHAPTER 8

Planes, coplanar lines, and solid angles

This chapter includes the first 21 propositions from Euclid's Book XI.

8.1. Definitions

1. When two or more lines are in one plane, they are said to be *coplanar*.

2. The angle which one plane makes with another is called a *dihedral angle*.

3. A *solid angle* is that which is made by more than two plane angles in different planes which meet a point.

4. The point from [Def. 7.3] is called the *vertex* of the solid angle.

5. If a solid angle is composed of three plane angles, it is called a *trihedral angle*; if of four, a *tetrahedral angle*; and if of more than four, a *polyhedral angle*.

6. A line which is perpendicular to a system of concurrent and coplanar lines is said to be perpendicular to the plane of these lines and is also called *normal* to it. (These lines will sometimes be called "normals" to a given plane. We may also have rays and segments which are normal to a plane.)

7. If from every point in a given line normals are drawn to a given plane, the locus of their feet is called the projection of the given line on the plane.

8. Two planes which meet are perpendicular to each other when the lines constructed perpendicular in one of them to their common section are normals to the other.

9. When two planes which meet are not perpendicular to each other, their inclination is the acute angle contained by two lines drawn from any point of their common section at right angles to it (one in one plane, and one in the other).

10. If at the vertex O of a trihedral angle O—ABC we construct normals OA, OB, OC to the faces OBC, OCA, OAB, respectively, in such a way that OA is on the same side of the plane OBC as OA, etc., the trihedral angle O—$A'B'C'$ is called the supplementary of the trihedral angle O—ABC.

8.2. Propositions from Book XI: 1-21

PROPOSITION 8.1. *If part of a line stands on a plane, then each part of that line must stand on that plane.*

PROOF. Construct the line AB on the plane X and cut AB at point C. We wish to show that BC is also on plane X.

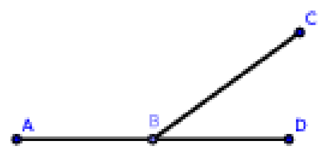


FIGURE 8.2.1. [7.1]

Since AB is on the plane X, it can be extended on X [Postulate 1.2]. Extend it to D. Then, if BC is not on X, let any other plane passing through AD be turned round AD until it passes through the point C. Because the points B, C are in this second plane, the line BC is on it. Therefore, the two lines ABC, ABD lying in one plane have a common segment AB, a contradiction. □

COROLLARY. *1. [7.1] holds for rays, mutatis mutandis.*

PROPOSITION 8.2. *Two segments which intersect one another at any point are coplanar as are any three segments which form a triangle.*

PROOF. We claim that two segments (AB, CD) which intersect one another at a point (E) are coplanar as are any three segments (EC, CB, BE) which form a triangle.

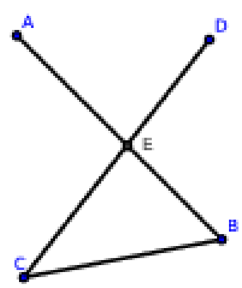


FIGURE 8.2.2. [7.2]

Let any plane pass through EB and be turned round it until it passes through C. Then because the points E, C are in this plane, the segment EC is in it [Def. 1.6]. For the same reason, the segment BC is on it. Therefore, EC, CB, BE are coplanar. Since AB and CD are two of these segments, AB and CD are coplanar. □

PROPOSITION 8.3. *If two planes cut one another, their intersection is a line.*

PROOF. We claim that if two planes (AB, BC) cut one another, their intersection is a line (BD).

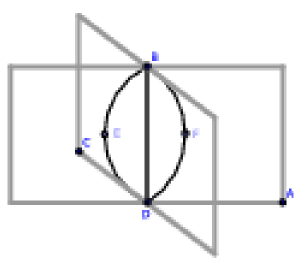


FIGURE 8.2.3. [7.3]

Otherwise, construct in the plane AB the line BED and in the plane BC construct the line BFD. Then the lines BED, BFD enclose a space, which contradicts [Axiom 1.10]. Therefore, the common section BD of the two planes is a line. □

PROPOSITION 8.4. *If a line is perpendicular to each of two intersecting lines, it will be perpendicular to any line which is both coplanar and concurrent with the intersecting lines.*

PROOF. If a line (EF) is perpendicular to each of two intersecting lines (AB, CD), it will be perpendicular to any line (GH) which is both coplanar and concurrent with them.

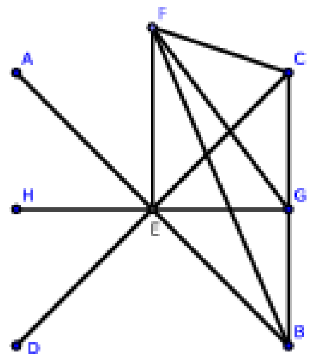


FIGURE 8.2.4. [7.4]

Through any point G in GH construct a line BC intersecting AB, CD which is bisected at G. Join any point F in EF to B, G, C. Then because $EF \perp EB$ and $EF \perp EC$, we have that

$$\begin{aligned} BF^2 &= BE^2 + EF^2 && \text{and} \\ CF^2 &= CE^2 + EF^2 && \Rightarrow \\ BF^2 + CF^2 &= BE^2 + CE^2 + 2EF^2 && \Rightarrow \end{aligned}$$

Also,

$$\begin{aligned} BE^2 + EF^2 &= 2BG^2 + 2GF^2 && \text{[2.10, Ex. 2], and} \\ BE^2 + CE^2 &= 2BG^2 + 2GE^2 && \Rightarrow \\ 2BG^2 + 2GF^2 &= 2BG^2 + 2GE^2 + 2EF^2 \\ GF^2 &= GE^2 + EF^2 \end{aligned}$$

Hence, the angle $\angle GEF$ is right, and so $EF \perp EG$. □

COROLLARY. *1. The normal is the least line that may be constructed from a given point to a given plane; of all others that may be constructed to it, the lines of any system making equal angles with the normal are equal to each other.*

COROLLARY. *2. A perpendicular to each of two intersecting lines is normal to their plane.*

PROPOSITION 8.5. *If three concurrent lines have a common perpendicular, they are coplanar.*

PROOF. If three concurrent lines (BC, BD, BE) have a common perpendicular (AB), they are coplanar.

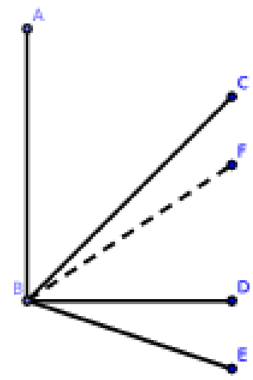


FIGURE 8.2.5. [7.5]

Suppose that BC is not coplanar with BD, BE, and let the plane of AB, BC intersect the plane of BD, BE at the line BF. By [7.3], BF is a segment; and, since it is coplanar with BD, we have that BE is perpendicular to AB (since each are perpendicular to AB, BF) [7.4]. Therefore, the angle $\angle ABF$ is right. We also have that the angle $\angle ABC$ is right by hypothesis. Hence, $\angle ABC = \angle ABF$, a contradiction [Axiom 1.9]. Therefore, the lines BC, BD, BE are coplanar. □

PROPOSITION 8.6. *TWO NORMAL LINES. If two lines are normals to the same plane, they are parallel to one another.*

PROOF. If two lines (AB, CD) are normals to the same plane (X), then $AB \parallel CD$.

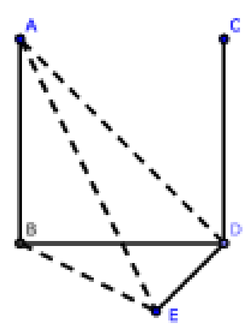


FIGURE 8.2.6. [7.6]

Let AB, CD meet the plane X at the points B, D. Join BD, and in the plane X construct $DE \perp BD$. Take any point E in DE. Join BE, AE, AD. Then because AB is normal to X, the angle $\angle ABE$ is right. Because the angle $\angle BDE$ is right, it follows that

$$AE^2 = AB^2 + BE^2 = AB^2 + BD^2 + DE^2$$

But $AB^2 + BD^2 = AD^2$ because the angle $\angle ABD$ is right. Hence $AE^2 = AD^2 + DE^2$. Therefore the angle $\angle ADE$ is right [1.48]. And since CD is normal to the plane X, $DE \perp CD$. Hence DE is a common perpendicular to the three concurrent lines CD, AD, BD. Therefore these lines are coplanar [7.5]. But AB is coplanar with AD, BD [7.2]. Therefore the lines AD, BD, CD are coplanar; and since the angles $\angle ABD$, $\angle BDC$ are right, we have that $AB \parallel CD$ [1.28]. □

Exercises.

1. The projection of any line on a plane is a straight line.

2. The projection on either of two intersecting planes of a normal to the other plane is perpendicular to the line of intersection of the planes.

PROPOSITION 8.7. *PARALLEL LINES AND THEIR INTERSECTIONS. Two parallel lines and any line intersecting them are coplanar.*

PROOF. Two parallel lines (AB, CD) and any line (EF) intersecting them are coplanar.

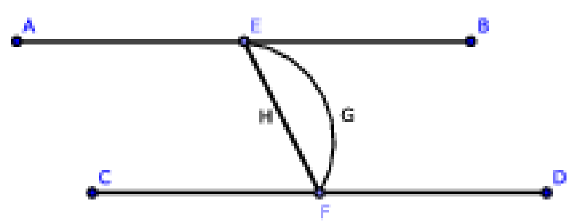


FIGURE 8.2.7. [7.7]

If possible, let the intersecting segment lie outside of the plane as segment EGF. In the plane, construct the segment EHF. Then we have two segments EGF, EHF enclosing a space, which contradicts [Axiom 1.10]. Hence, the two parallel straight lines and the transversal are coplanar. □

Alternatively:

PROOF. Since the points E, F are in the plane of the parallels, the segment joining these points also lies in that plane. □

PROPOSITION 8.8. *NORMAL PARALLEL LINES. If one of two parallel straight lines is normal to a plane, the other line is normal to the same plane.*

PROOF. If one of two parallel straight lines (AB, CD) is normal to a plane (X), then the other line is normal to the same plane.

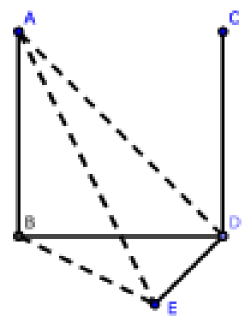


FIGURE 8.2.8. [7.8]

Let AB, CD meet in the plane X at the points B, D. Join BD. Then the lines AB, BD, CD are coplanar. Wlog, suppose that AB is normal to the X. Construct $DE \perp BD$. Take any point E in DE and join BE, AE, AD. Then because AB is normal to the plane X, it is perpendicular to the line BE in that plane [Def 7.6].

Hence, the angle $\angle ABE$ is right, and so

$$AE^2 = AB^2 + BE^2 = AB^2 + BD^2 + DE^2$$

(see the proof of [7.6]). Therefore, the angle $\angle ADE$ is right. Hence, DE is at right angles both to AD and BD. Therefore $DE \perp CD$ [7.4], and DE is coplanar and concurrent with AD and BD.

Again, since $AB \parallel CD$, $\angle ABD + \angle BDC$ is two right angles [1.29]. Since $\angle ABD$ is right by hypothesis, it follows that $\angle BDC$ is also right. Hence CD is perpendicular to the two lines DB, DE, and therefore it is normal to the plane X by [7.4]. □

PROPOSITION 8.9. *TRANSITIVITY OF PARALLEL LINES. Two lines which are each parallel to a third line are also parallel to one another.*

PROOF. Two lines (AB, CD) which are each parallel to a third line (EF) are also parallel to one another.

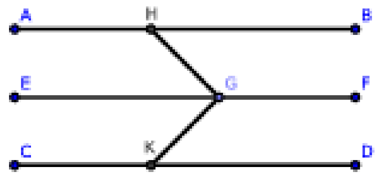


FIGURE 8.2.9. [7.9]

If the three lines are coplanar, the proposition is evidently the same as [1.30].

Otherwise, from any point G in EF, construct in the planes of EF, AB and EF, CD (respectively) the lines GH, GK where each is perpendicular to EF [1.11]. Because EF is perpendicular to each of the lines GH, GK, it is normal to their plane [7.4]. And because $AB \parallel EF$ by hypothesis and EF is normal to the plane GHK, AB is normal to the plane GHK [7.8]. Similarly, CD is normal to the plane HGK. Hence, since AB and CD are normals to the same plane, they are parallel to one another. □

PROPOSITION 8.10. *ANGLES AND PARALLEL LINES. If two intersecting lines are respectively parallel to two other intersecting straight lines, the angle between the former is equal to the angle between the latter.*

PROOF. If two intersecting lines (AB, BC) are respectively parallel to two other intersecting lines (DE, EF), the angle ($\angle ABC$) between the former is equal to the angle ($\angle DEF$) between the latter.

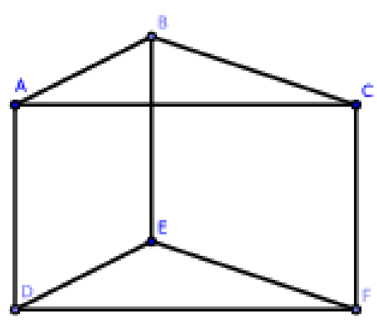


FIGURE 8.2.10. [7.10]

If both pairs of lines are coplanar, the proposition is the same as [1.29, #2].

Otherwise, take any points A, C in the lines AB, BC and cut off $ED = BA$, and $EF = BC$ 1.3]. Join AD, BE, CF, AC, DF. Then because AB is equal and parallel to DE, AD is equal and parallel to BE [1.33].

Similarly, $CF = BE$ and $CF \perp BE$. Hence, $AD = CF$, $AD \parallel CF$ [7.9], and $AC = DF$ [1.33]. Therefore, the triangles $\triangle ABC$, $\triangle DEF$ have the three sides of one respectively equal to the three sides of the other. By [1.8], $\angle ABC = \angle DEF$. □

PROPOSITION 8.11. *CONSTRUCTION OF A NORMAL LINE I. We wish to construct a normal to a given plane from a given point not in the plane.*

PROOF. We wish to construct a normal to a given plane (BH) from a given point (A) not in the plane.

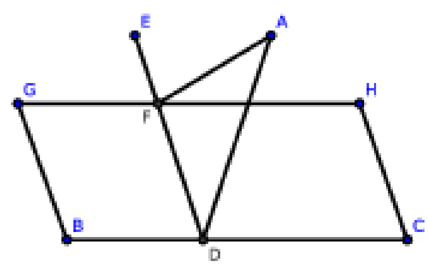


FIGURE 8.2.11. [7.11]

In the given plane BH construct any line BC, and from A construct $AD \perp BC$ [1.12]; if AD is perpendicular to the plane, the proof follows.

Otherwise, from D construct DE in the plane BH at right angles to BC [1.11] and from A construct $AF \perp DE$ [1.12]. We claim that AF is normal to the plane BH.

To see this, construct $GH \parallel BC$. Because BC is perpendicular both to ED and DA, it is normal to the plane of ED, DA [11.4]. And since $GH \parallel BC$, it is normal to the same plane [11.8]. Hence $AF \perp GH$ [Def. 11.6], and $AF \perp DE$ by construction. Therefore, AF is normal to the plane of GH and ED, which is the plane BH. □

PROPOSITION 8.12. *CONSTRUCTION OF A NORMAL LINE II. Construct a normal to a given plane from a given point in the plane.*

PROOF. We wish to construct a normal to a given plane from a given point (A) in the plane.

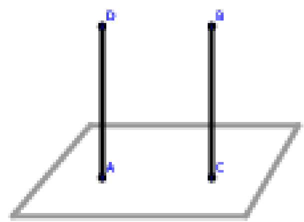


FIGURE 8.2.12. [7.12]

From any point B not in the plane construct BC normal to it [11.11]. If this line passes through A, it is the normal required.

Otherwise, from A construct $AD \parallel BC$ [1.32]. Because $AD \parallel BC$ and BC is normal to the plane, AD is also normal to the plane [11.8], and it is drawn from the given point. Hence, the proof. □

PROPOSITION 8.13. *UNIQUENESS OF NORMAL LINES. From a given point, there exists a unique normal to a given plane.*

PROOF. From a given point (A), we claim that there exists a unique normal to a given plane (X).

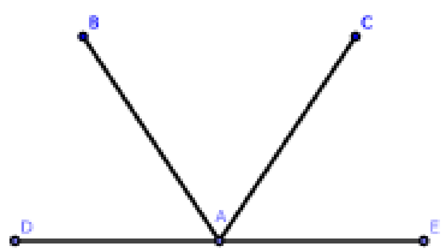


FIGURE 8.2.13. [7.13]

We shall prove this claim in two parts:

1. Let A is in the given plane and suppose that AB, AC are both normals to it on the same side. Let the plane of BA, AC cut the given plane X at the line DE. Because BA is a normal, the angle $\angle BAE$ is right. Similarly, $\angle CAE$ is right. Hence $\angle BAE = \angle CAE$, a contradiction.

2. If the point A is above the plane, there can exist only one normal; otherwise, the two would be parallel to one another [11.6], a contradiction.

Hence, the proof. □

PROPOSITION 8.14. *PARALLEL PLANES. Planes which have a common normal are parallel to each other.*

PROOF. We claim that planes (CD, EF) which have a common normal (AB) are parallel to each other.

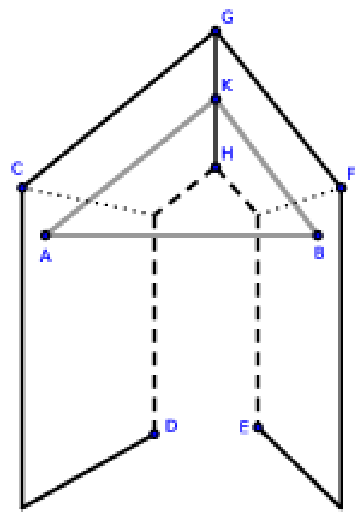


FIGURE 8.2.14. [7.14]

Let GH be the line of intersection between planes CD, EF. Join AK, BK. Because AB is normal to the plane CD, $AB \perp AK$, which it meets in that plane [Def. 11.6], the angle $\angle BAK$ is right. Similarly, the angle $\angle ABK$ is right, and the plane triangle $\triangle ABK$ has two right angles, a contradiction. Since the planes CD, EF do not intersect, they are parallel. □

Exercises.

1. The angle between two planes is equal to the angle between two intersecting normals to these planes.

2. If a line is parallel to each of two planes, the sections which any plane passing through the line makes with the planes are parallel.

3. If a line is parallel to each of two intersecting planes, it is parallel to their intersection.

4. If two lines are parallel, they are parallel to the common section of any two planes passing through them.

5. If the intersections of several planes are parallel, the normals drawn to them from any point are coplanar.

PROPOSITION 8.15. *CHARACTERISTIC OF PARALLEL PLANES. Two planes are parallel if two intersecting lines on one plane are respectively parallel to two intersecting lines on the other plane.*

PROOF. Two planes (AC, DF) are parallel if two intersecting lines (AB, BC) on one plane are respectively parallel to two intersecting lines (DE, EF) on the other plane.

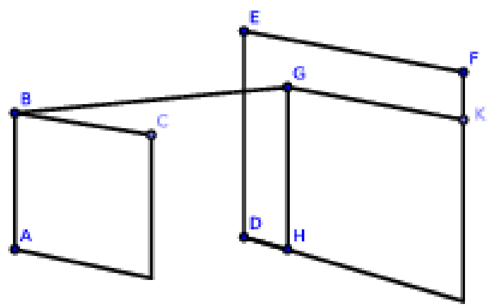


FIGURE 8.2.15. [7.15]

From B construct BG perpendicular to the plane DF [11.11] and let it intersect plane DF at point G. Through G construct $GH \parallel ED$ and $GK \parallel EF$. Since $GH \parallel ED$ by construction and $AB \parallel ED$ by hypothesis, $AB \parallel GH$ [11.9]. Hence, $\angle ABG + \angle BGH$ equals two right angles [1.29]. Since $\angle BGH$ is a right angle by construction, $\angle ABG$ is also right. Similarly, $\angle CBG$ is right. Hence BG is normal to the plane AC [Def. 11.6] as well as normal to DF by construction. Hence the planes AC, DF have a common normal BG; therefore, they are parallel to one another. □

PROPOSITION 8.16. *PARALLEL PLANES AND AN INTERSECTING PLANE. If two parallel planes are cut by a third plane, their common sections with the third plane are parallel.*

PROOF. If two parallel planes (AB, CD) are cut by a third plane (FG), their common sections with the third plane (EF, GH) are parallel.

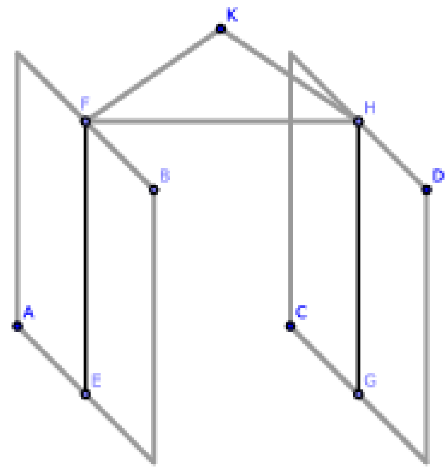


FIGURE 8.2.16. [7.16]

If the lines EF, GH are not parallel, they must meet at some finite distance. Let them meet at K. Since K is a point on the line EF and EF is on the plane AB, K is in the plane AB. Similarly, K is a point on the plane CD. Hence, the planes AB, CD meet at K, a contradiction since they are parallel. Therefore, the lines EF, GH are parallel. □

Exercises.

1. Parallel planes intercept equal segments on parallel lines.

2. Parallel lines intersecting the same plane make equal angles with that plane.

3. A straight line intersecting parallel planes makes equal angles with he parallel planes.

PROPOSITION 8.17. *PROPORTIONAL AND PARALLEL LINES. If two parallel lines are cut by three parallel planes in two triads of points, their segments between those points are proportional.*

PROOF. If two parallel lines (AB, CD) are cut by three parallel planes (GH, KL, MN) in two triads of points (A, E, B and C, F, D where A, E, B and C, F, D are respectively collinear), then the segments between those points are proportional; or, $AE : EB :: CF : FD$.

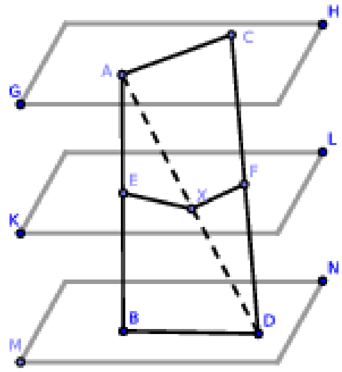


FIGURE 8.2.17. [7.17]

Join AC, BD, AD. Let AD meet the plane KL at point X. Join EX, XF. Because the parallel planes KL, MN are cut by the plane ABD at the lines EX, BD, these lines are parallel [11.16]. Hence $AE : EB :: AX : XD$ [6.2]. Similarly, $AX : XD :: CF : FD$. By [5.11], it follows that $AE : EB :: CF : FD$. □

PROPOSITION 8.18. *TRANSITIVITY OF PERPENDICULAR PLANES. If a line is normal to a plane, any plane passing through the line is perpendicular to that plane.*

PROOF. If a line (AB) is normal to a plane (CI), we claim that any plane (DE) passing through the line is perpendicular to CI.

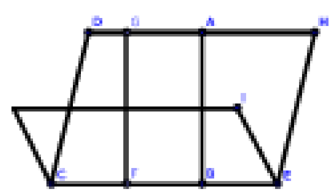


FIGURE 8.2.18. [9.18]

Let CE be the common section of the planes DE, CI. From any point F on CE, construct FG in the plane DE such that $FG \parallel AB$ [1.31]. Now $AB \parallel FG$ and AB is normal to the plane CI; hence, FG is normal to CI [11.8]. Since $FG \parallel AB$, we have that $\angle ABF + \angle BFG$ are equal to two right angles [1.29]. Since $\angle ABF$ is right by hypothesis, $\angle BFG$ is right and therefore $FG \perp CE$. Hence every line in the plane DE drawn perpendicular to the common section of the planes DE, CI is normal to the plane CI. Therefore by [Def. 8.11], the planes DE, CI are perpendicular to each other. □

PROPOSITION 8.19. *INTERSECTING PLANES. If two intersecting planes are each perpendicular to a third plane, their common section is normal to that plane.*

PROOF. If two intersecting planes (AB, BC) are each perpendicular to a third plane (ADC), their common section (BD) is normal to ADC.

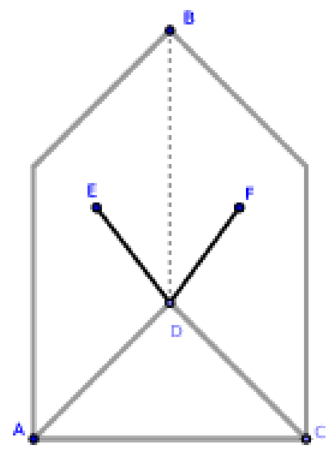


FIGURE 8.2.19. [7.19]

Otherwise, construct the line DE from D in the plane AB such that $DE \perp AD$ where AD is the common section of the planes AB, ADC. In the plane BC, construct BF perpendicular to the common section DC of the planes BC, ADC. Because the plane AB is perpendicular to ADC, the line DE in AB is normal to the plane ADC [Def. 7.8]. Similarly, DF is normal to it. Therefore from the point D there are two distinct normals to the plane ADC, a contradiction [7.13]. Hence, BD is normal to the plane ADC. □

Exercises.

1. If three planes have a common line of intersection, the normals drawn to these planes from any point of that line are coplanar.

2. If two intersecting planes are respectively perpendicular to two intersecting lines, the line of intersection of the former is normal to the plane of the latter.

3. In the last case, show that the dihedral angle between the planes is equal to the rectilinear angle between the normals.

PROPOSITION 8.20. *TRIHEDRAL ANGLES. The sum of any two plane angles of a trihedral angle is greater than the third.*

PROOF. The sum of any two plane angles ($\angle BAD$, $\angle DAC$) of a trihedral angle (at point A) is greater than the third ($\angle BAC$).

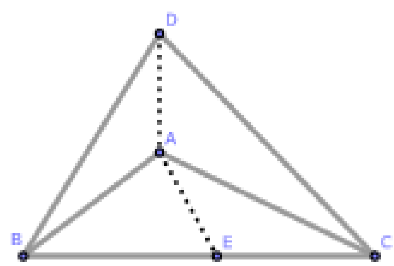


FIGURE 8.2.20. [7.20]

If the third angle $\angle BAC$ is less than or equal to either of the other angles, the proposition is evident.

Otherwise, suppose it greater: take any point D in AD and at the point A in the plane BAC make $\angle BAE = \angle BAD$ [1.23]. Cut off $AE = AD$. Through E construct BC, cutting AB, AC at the points B, C. Join DB, DC. Then the triangles $\triangle BAD$, $\triangle BAE$ have the two sides BA, AD in one equal respectively to the two sides BA, AE in the other and $\angle BAD = \angle BAE$. Therefore the third side $BD = BE$.

However, the sum of the sides BD, DC is greater than BC; hence DC is greater than EC. Again, because the triangles $\triangle DAC$, $\triangle EAC$ have the sides DA, AC respectively equal to the sides EA, AC in the other where the base DC greater than EC. By [1.25], the angle $\angle DAC$ is greater than $\angle EAC$, but $\angle DAB = \angle BAE$ by construction. Hence, $\angle BAD + \angle DAC > \angle BAC$. □

PROPOSITION 8.21. *SUM OF PLANE ANGLES. The sum of all the plane angles forming any solid angle is less than four right angles.*

PROOF. The sum of all the plane angles ($\angle BAC$, $\angle CAD$, etc.) forming any solid angle (at A) is less than four right angles.

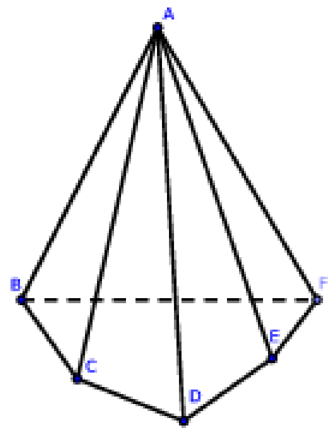


FIGURE 8.2.21. [7.21]

Suppose for the sake of simplicity that the solid angle at A is contained by five plane angles $\angle BAC$, $\angle CAD$, $\angle DAE$, $\angle EAF$, $\angle FAB$. Let the planes of these angles be cut by another plane at the lines BC, CD, DE, EF, FB. By [11.20], we have $\angle ABC + \angle ABF$ greater in measure than $\angle FBC$, $\angle ACB + \angle ACD$ greater in measure than $\angle BCD$, etc.

Adding these, we obtain the sum of the base angles of the five triangles $\triangle BAC$, $\triangle CAD$, etc., greater than the sum of the interior angles of the pentagon $BCDEF$; that is, greater than six right angles.

But the sum of the base angles of the same triangles, together with the sum of the plane angles $\angle BAC$, $\angle CAD$, etc., forming the solid angle at A is equal to twice as many right angles as there are triangles $\triangle BAC$, $\triangle CAD$, etc.; that is, equal to ten right angles. Hence, the sum of the angles forming the solid angle is less than four right angles. □

Observation: this proposition may not hold if the polygonal base $BCDEF$ contains re-entrant angles.

Chapter 7 exercises.

1. Any face angle of a trihedral angle is less than the sum (but greater than the difference) of the supplements of the other two face angles.

2. A solid angle cannot be formed of equal plane angles which are equal to the angles of a regular polygon of n sides except when $n = 3, 4, 5$.

3. Through one of two non-coplanar lines, construct a plane parallel to the other.

4. Construct a common perpendicular to two non-coplanar lines and show that it is the shortest distance between them.

5. If two of the plane angles of a tetrahedral angle are equal, the planes of these angles are equally inclined to the plane of the third angle, and conversely. If two of the planes of a trihedral angle are equally inclined to the third plane, the angles contained in those planes are equal.

6. Prove that the three lines of intersection of three planes are either parallel or concurrent.

7. If a trihedral angle O is formed by three right angles and A, B, C are points along the edges, the orthocenter of the triangle $\triangle ABC$ is the foot of the normal from O on the plane ABC.

8. If through the vertex O of a trihedral angle O—ABC any line OD is drawn interior to the angle, the sum of the rectilinear angles $\angle DOA$, $\angle DOB$, $\angle DOC$ is less than the sum but greater than half the sum of the face angles of the trihedral.

9. If on the edges of a trihedral angle O—ABC three equal segments OA, OB, OC are taken, each of these is greater than the radius of the circle described about the triangle $\triangle ABC$.

10. Given the three angles of a trihedral angle, find by a plane construction the angles between the containing planes.

11. If any plane P cuts the four sides of a Gauche quadrilateral $ABCD$ (a quadrilateral whose angular points are not coplanar) at four points, a, b, c, d, then given the four ratios

$$\frac{Aa}{aB}, \frac{Bb}{bC}, \frac{Cc}{cD}, \frac{Dd}{dA}$$

we have that

$$\frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = 1$$

Conversely, if

$$\frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = \pm 1$$

then the points a, b, c, d are coplanar.

12. If in #11 the intersecting plane is parallel to any two sides of the quadrilateral, it cuts the two remaining sides proportionally.

13. If O—$A'B'C'$ is the supplementary of O—ABC, prove that O—ABC is the supplementary angle of O—$A'B'C'$.

14. If two trihedral angles are supplementary, each dihedral angle of one is the supplement of the corresponding face angle of the other.

15. Through a given point, construct a line which will meet two non-coplanar lines.

16. Construct a line parallel to a given line which will meet two non-coplanar lines.

17. Given an angle $\angle AOB$, prove that the locus of all the points P of space where the sum of the projections of the line OP on OA and OB are constant is a plane.

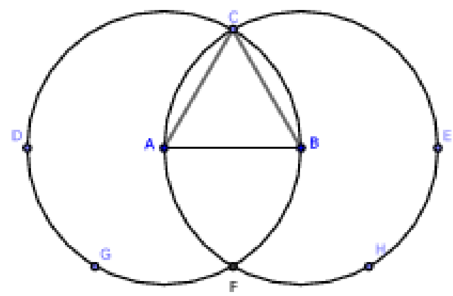
Part 3

Student Answer Key

CHAPTER 1

Solutions: Angles, Parallel Lines, Parallelograms

[1.1] Exercises



The following two exercises use Fig. 1.5.1 (above) and are to be solved when the student has completed Chapter 1.

1. If the segments AF, BF are joined, prove that the figure $\boxdot ACBF$ is a lozenge.

PROOF. Suppose that AF, BF are joined. By an argument similar to the proof of [1.1], we have that $AB = AF = BF$. By [1.1], we have that $AC = AB = BC$. Hence, it follows that

$$AC = BC = BF = AF$$

and so by [Def 1.29], $\boxdot ACBF$ is a lozenge. $\square$

---

2. If AB is extended to the circumferences of the circles (at points D and E), prove that the triangles $\triangle CDF$ and $\triangle CEF$ are equilateral.

PROOF. Construct segments CD, DF, FE, CE, and CF as per the hypothesis. Also extend AB to a line. We wish to show that $\triangle CDF$ and $\triangle CEF$ are equilateral.

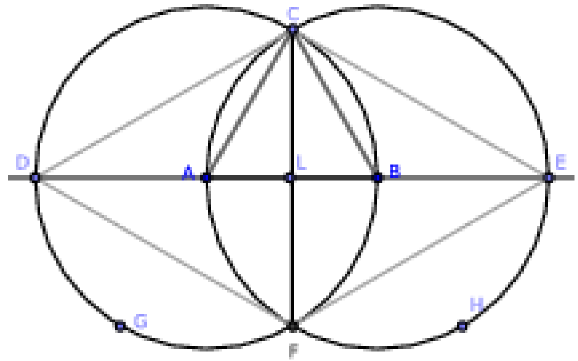


FIGURE 1.0.1. [1.1, #2]

By [1.1], $\angle ACB = \angle BAC = \angle ABC$. [1.32, Cor. 6] states that each angle of an equilateral triangle equals two-thirds of a right angle; since a right angle equals $\frac{\pi}{2}$ radians, $\angle ACB = \angle BAC = \angle ABC = \frac{\pi}{3}$.

Since $\boxdot ACBF$ is a lozenge by [1.1, #1], CF is its Axis of Symmetry. Hence, $\angle ACL = \frac{\pi}{6}$. Consider $\triangle ACL$. It follows that $\angle ALC = \frac{\pi}{2}$. Similarly, we can show that in $\triangle BCL$ $\angle BLC = \frac{\pi}{2}$, $\angle BCL = \frac{\pi}{6}$, and $\angle LBC = \frac{\pi}{3}$. Since $\triangle ACL$ and $\triangle BCL$ share side CL, by [1.26] we have that $\triangle ACL \cong \triangle BCL$.

Consider $\triangle DAC$. Notice that $\angle DAC = \frac{2\pi}{3}$ since $\angle DAC$ and $\angle LAC$ are supplementary. Since AC and AD are both radii of $\circ CDF$, we have that $\angle ADC = \angle ACD = \frac{\pi}{6}$ [1.5]. Hence in $\triangle DCL$, $\angle LDC = \frac{\pi}{6}$ and $\angle DCL = \frac{\pi}{3}$. Similarly, we can show that in $\triangle ECL$ that $\angle LEC = \frac{\pi}{6}$ and $\angle LCE = \frac{\pi}{3}$. Since $\triangle DCL$ and $\triangle ECL$ share side CL, by [1.26] we have that $\triangle DCL \cong \triangle ECL$. It follows that

$$DC = CE$$

Similarly we can show that $\triangle DFL \cong \triangle EFL$, and so $DF = FE$. This will also show that $\angle FDL = \angle CDL$, $\angle FLD = \angle CLD$. Since $\triangle CDL$ and $\triangle FDL$ share side DL, by [1.26], $\triangle CDL \cong \triangle FDL$, so $DC = DF$. Hence,

$$DC = CE = DF = FE$$

Finally, we have that $\angle FDC = \angle FEC$, $\angle DCF = \angle ECF$, and $\triangle DCF$ and $\triangle ECF$ share side CF. By [1.26], $\triangle CDF \cong \triangle CEF$. Since each angle of each triangle equals $\frac{\pi}{3}$ radians, the triangles are each equilateral. □

COROLLARY. *$\boxdot CEFD$ is a lozenge.*

[1.2] Exercises

1. Prove [1.2] when A is a point on BC.

PROOF. Let BC be an arbitrary segment such that A is a point on BC. If $A = B$ or $A = C$, the proof follows trivially.

Suppose that A is not an endpoint of BC. Construct the equilateral triangle $\triangle ABD$. Also construct the circle $\circ CEF$ with center A and radius equal in length to AC. Extend side DA to the point E on the circumference of $\circ CEF$.

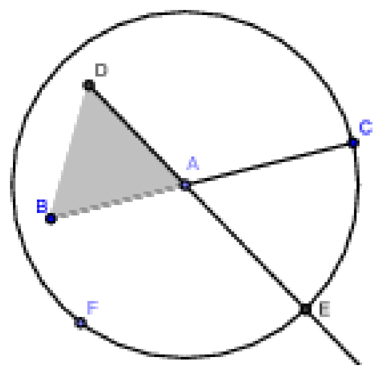


FIGURE 1.0.2. [1.2, #1]

Since $\triangle ABD$ is equilateral, $AB = AD$. Since AC and AE are radii of $\circ CEF$, $AC = AE$. Hence,

$$DE = AD \oplus AE = BC$$

By using [Axiom 1.8] (specifically the principle of superposition), we may move DE so that the point D coincides with point A. The proof follows. □

[1.4] Exercises

Prove the following:

1. The line that bisects the vertical angle of an isosceles triangle bisects the base perpendicularly.

PROOF. Suppose that $\triangle ABC$ is an isosceles triangle (where $AB = AC$). Further suppose that the ray AD bisects the angle $\angle BAC$. (A line, ray, or a segment of appropriate size may be used in this proof, *mutatis mutandis*.) We wish to show that $\angle ADB = \angle ADC$ and that $BD = CD$.

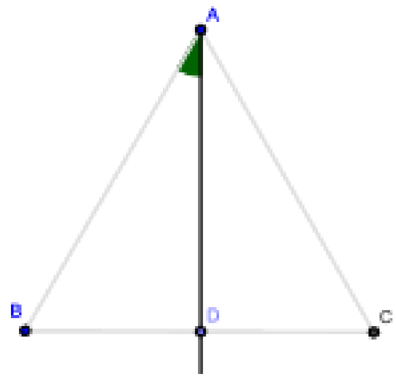


FIGURE 1.0.3. [1.4, #1]

Since $AB = AC$ and $\angle DAB = \angle DAC$ by hypothesis, and since $\triangle ABD$ and $\triangle ACD$ share the side AD, by [1.4] we have that

$$\triangle ABD \cong \triangle ACD$$

Hence, $BD = CD$. Also, $\angle ADB = \angle ADC$. Since the $\angle ADB$ and $\angle ADC$ are supplements, they are right angles by [Def. 1.14]; hence, AD is a perpendicular bisector of the base (AC) of $\triangle ABC$. □

---

2. If two adjacent sides of a quadrilateral are equal and the diagonal bisects the angle between them, then their remaining sides are equal.

PROOF. Suppose that $ABCD$ is a quadrilateral where $AB = AC$ and where the diagonal AD bisects $\angle BAC$. We wish to show that $BD = CD$.

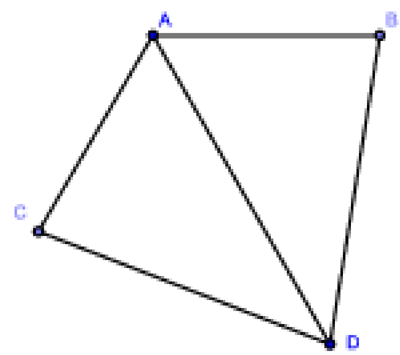


FIGURE 1.0.4. [1.4, #2]

Since $AC = AB$ and $\angle CAD = \angle DAB$ by hypothesis, and since $\triangle ACD$ and $\triangle ABD$ share the side AD, by [1.4] we have that

$$\triangle ACD \cong \triangle ABD$$

Hence, $BD = CD$. □

---

3. If two segments stand perpendicularly to each other and if each bisects the other, then any point in either segment is equally distant from the end-points of the other segment.

PROOF. Suppose that segments AB and CD stand perpendicularly to each other and bisect each other at point E. Let F be a point on AB. We claim that F is equally distant from C and D.

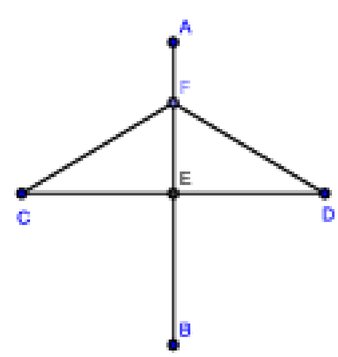


FIGURE 1.0.5. [1.4, #3]

Construct $\triangle CEF$ and $\triangle DEF$. Since AB perpendicularly bisects CD, $\angle CEF = \angle DEF$ and $CE = DE$. Since $\triangle CEF$ and $\triangle DEF$ share side FE, by [1.4] we have that

$$\triangle CEF \cong \triangle DEF$$

Hence, $CF = DE$. The proof for a point on CD is similar to the above, *mutatis mutandis*. □

[1.5] Exercises

2. Prove that the line joining the point A to the intersection of the segments CF and BG is an Axis of Symmetry of $\triangle ABC$.

PROOF. Construct the line AH on the figure from [1.5] where H is in the intersection of the segments CF and BG. We wish to show that AH is the Axis of Symmetry of $\triangle ABC$.

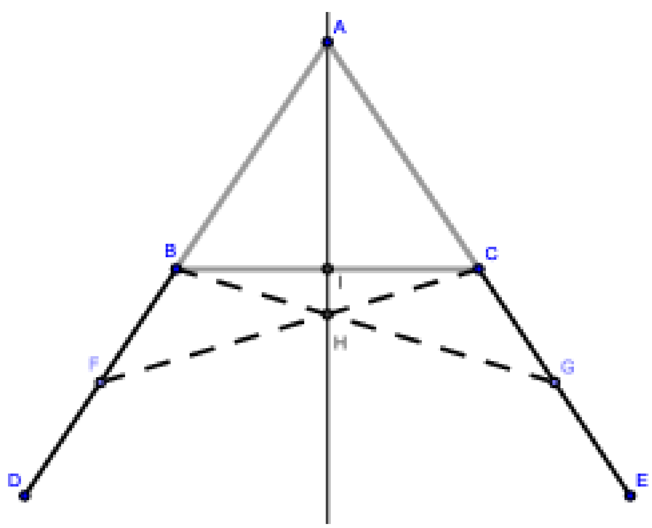


FIGURE 1.0.6.

By [1.5], we have that $\triangle FBC \cong \triangle GCB$. Subtracting $\triangle HBC$ from each, we have that $\triangle FBH \cong \triangle GCH$. Hence, $HB = HC$. Since F was chosen arbitrarily, the position of H is arbitrary in the proof of [1.5], and so we have that the distance from B to any point on the line AH is equal to the distance from C to that point.

Let I be the point on AH which intersects the base of $\triangle ABC$. By the above, $BI = CI$. By [Def. 1.35], AH is the Axis of Symmetry of $\triangle ABC$. □

6. If three points are taken on the sides of an equilateral triangle (one on each side and at equal distances from the angles), then the segments joining them form a new equilateral triangle.

PROOF. Suppose that $\triangle ABC$ is equilateral. Construct points I, J, and K on sides BC, AB, and AC, respectively. Since each point on each side is at an equal distance from the endpoints of the relevant side, each point is the midpoint of the side on which it stands. Construct We wish to show that $\triangle IJK$ is equilateral.

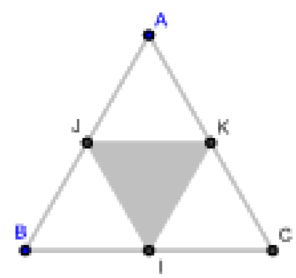


FIGURE 1.0.7.

Since I is the midpoint of side BC, $CI = BI$. Since J is the midpoint of side AB, $JA = JB$; since $\triangle ABC$ is equilateral, $IB = JB$. Continuing in this manner, we can show that

$$IB = JB = JA = AK = KC = IC$$

And since $\triangle ABC$ is equilateral, by [1.5, Cor. 1] we also have that

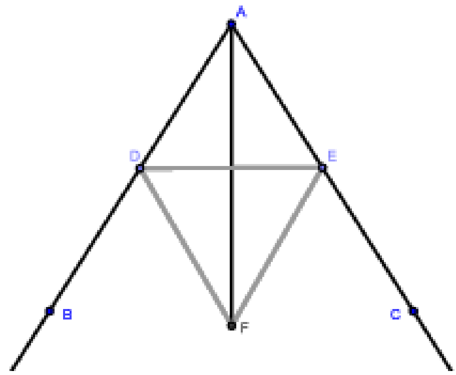
$$\angle ABC = \angle ACB = \angle BAC$$

Hence by [1.4],

$$\triangle JBI \cong \triangle KCI \cong \triangle JAK$$

It follows that $IJ = JK = KI$, and so $\triangle IJK$ is equilateral. □

[1.9] Exercises



2. Prove that $AF \perp DE$. (Hint: the proof follows almost immediately from [1.5, #2].)

PROOF. Extend AF to a line. By [1.5, #2], AF is the Axis of Symmetry of $\triangle ADE$. Hence, if H is the intersection of DE and AF, then $DH = HE$. Consider $\triangle ADH$ and $\triangle AEH$. Since each triangle shares side AH and $DA = EA$ by construction, by [1.8] we have that $\triangle ADH \cong \triangle ADE$. Hence, $\angle DHA = \angle EHA$. By [Def. 1.14], $\angle DHA$ and $\angle EHA$ are right angles. Hence, $AF \perp DE$. □

---

3. Prove that any point on AF is equally distant from the points D and E.

PROOF. By [1.5, #2], AF is an Axis of Symmetry of $\triangle ABC$. The proof of this problem follows immediately by the proof of [1.5, #2]. □

[1.10] Exercises

1. Bisect a segment by constructing two circles.

PROOF. Construct the figure from [1.1]. We wish to bisect segment AB.

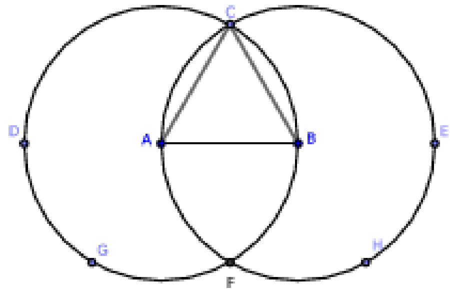


FIGURE 1.0.8. [1.10, #1]

If we construct a segment from C which intersects AB and bisects $\angle ACB$, the proof follows immediately from the proof of [1.10]. □

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2. Extend CD to a line. Prove that every point equally distant from the points A, B are points in the line CD.

PROOF. Construct the figure from [1.10] and extend CD to a line.

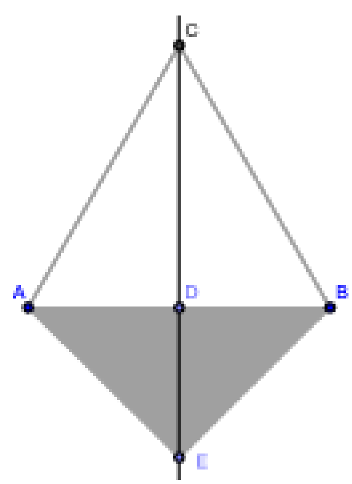
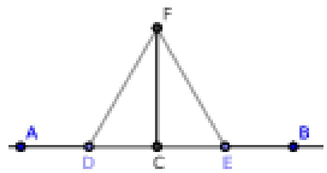


FIGURE 1.0.9. [1.10, #2]

By [1.9, Cor. 1], the line CD is an Axis of Symmetry to AB. If point E is an equal distance from both points A and B, then E must lie on the Axis of Symmetry of AB, which is CD □

[1.11] Exercises



1. Prove that the diagonals of a lozenge bisect each other perpendicularly.

PROOF. Construct the figure from [1.11], and also construct the equilateral triangle $\triangle DEG$ where G lies on the opposite side of AB from the point F. Construct the segment GC. By a similar argument to the proof of [1.11], $GC \perp AB$.

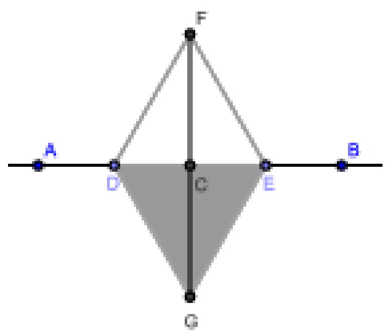


FIGURE 1.0.10. [1.11, #1]

Consider the triangles $\triangle DCF$ and $\triangle DCG$. Since $DF = DG$ by construction, the triangles share side DC, and $FC = GC$ by [1.47], we have by [1.8] that $\triangle DCF \cong \triangle DCG$. Similarly, it follows that

$$\triangle DCF \cong \triangle DCG \cong \triangle ECF \cong \triangle ECG$$

Hence,

$$FD = FE = GD = GE$$

and so $\boxdot FEGD$ is a lozenge.

Clearly, $GC \oplus CF = GF$ where GF is a diagonal of $\boxdot DFEG$; similarly, DE is the other diagonal of $\boxdot DFEG$. By the above, $GF \perp DE$. Hence, the proof. □

3. Find a point on a given line that is equally distant from two given points.

PROOF. Suppose that AB is our given line and that C and D are our given points. We wish to find a point F on AB which is equally distant from C and D.

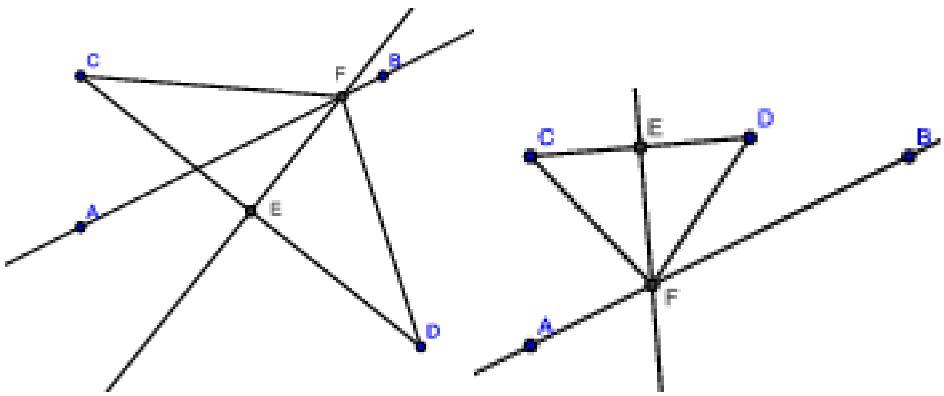


FIGURE 1.0.11. [1.11, #3]

Construct the segment CD and by [1.10] locate its midpoint, E. Construct FE where $CD \perp FE$ and F is a point on AB. We claim that F is the required point. Consider $\triangle CEF$ and $\triangle DEF$. $CE = DE$ by construction, $\angle CEF = \angle DEF$ by construction, and the triangles share side EF. By [1.4], $\triangle CEF \cong \triangle DEF$. Hence, $CF = DF$. □

5. Find a point that is equidistant from three given points. (Hint: you are looking for the circumcenter of the triangle formed by the points.)

PROOF. Construct three arbitrary points A, B, and C. Connect the three points, constructing $\triangle ABC$. By [1.10], we may locate the midpoints of each side of the triangle where the midpoint of AC is F, the midpoint of BC is E, and the midpoint of AB is D. Construct the line FG such that $FG \perp AC$. Similarly, construct the line EG such that $EG \perp BD$ and line DG such that $DG \perp AB$. Since none of the pairs of sides of $\triangle ABC$ are parallel, the lines FG, DG, and EG have a common point of intersection: G. We claim that G is equidistant from A, B, and C.

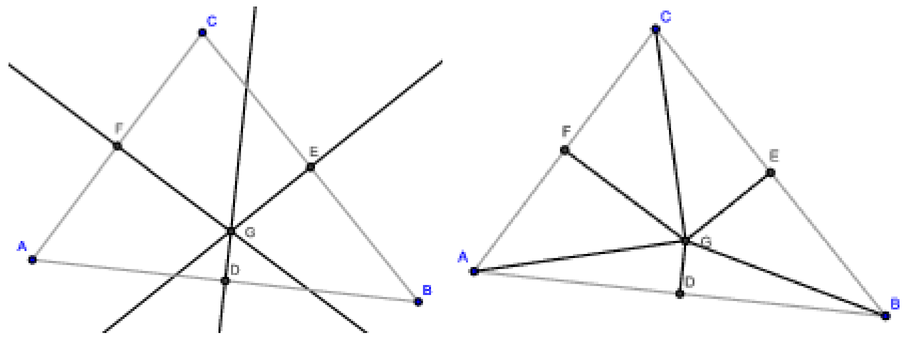


FIGURE 1.0.12. [1.10, #5]

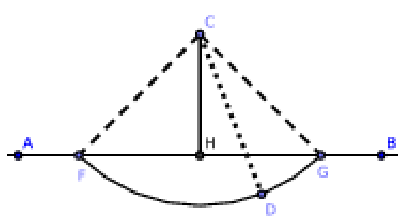
Construct the segments AG, BG, and CG. Consider $\triangle ADG$ and $\triangle BDG$. Since $AD = BD$, $AB \perp DG$, and the triangles share side DG, by [1.4], $\triangle ADG \cong \triangle BDG$. Hence, $AG = BG$.

Similarly, consider $\triangle AFG$ and $\triangle CFG$. By an argument similar to the above, we have that $\triangle AFG \cong \triangle CFG$, and so $AG = CG$. It follows that

$$AG = BG = CG$$

and hence G is equidistant from points A, B, C. □

[1.12] Exercises



1. Prove that circle $\circ FDG$ cannot meet AB at more than two points.

PROOF. Suppose that $\circ FDG$ intersects AB at more than two points. If the third point lies between points F and G, then the radius of $\circ FDG$ must decrease in length, a contradiction, since a circle's radius is a fixed length. Similarly, if the third point lies to the left of F or to the right of G, the radius of $\circ FDG$ must increase in length, also a contradiction.

Hence, $\circ FDG$ cannot meet AB at more than two points. □

[1.19] Exercises

3. Prove that three equal segments cannot be constructed from the same point to the same line.

PROOF. Construct the line AB and point C. Construct the segments CA, CB, and CD such that $CA = CB$. We claim that CD cannot be constructed such that $CA = CB = CD$.

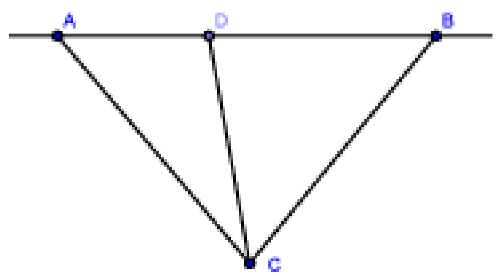


TABLE 1. [1.19, #3]

Suppose that $CA = CB = CD$. By [1.19, Cor. 1], $\angle CDA = \angle CAD$ and $\angle CAB = \angle CBA$. Hence, $\angle CDA = \angle CBD$; but by [1.16], we have that $\angle CDA > \angle CBD$, a contradiction.

A similar result occurs if point D does not fall between points A and B, *mutatis mutandis*. Hence, the proof. □

[1.20] Exercises

5. The perimeter of a quadrilateral is greater than the sum of its diagonals.

PROOF. Suppose that $ABCD$ is a quadrilateral with diagonals AC and BD. We wish to show that

$$AB + BC + CD + DA > AC + BD$$

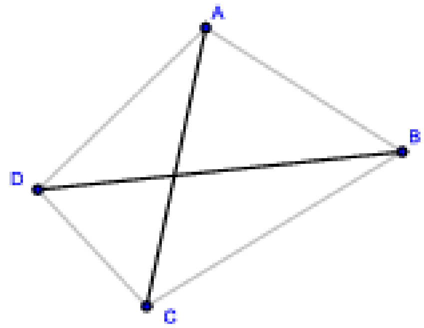


FIGURE 1.0.13. [1.20, #5]

By [1.20], we have that

$$\begin{aligned} AD + DC &> AC \\ AB + BC &> AC \\ AD + AB &> BD \\ BC + CD &> BD \end{aligned}$$

Or,

$$\begin{aligned} AB + BC + CD + DA &> 2 \cdot AC \\ AB + BC + CD + DA &> 2 \cdot BD \\ &\Longrightarrow \\ 2(AB + BC + CD + DA) &> 2(AC + BD) \\ &\Longrightarrow \\ AB + BC + CD + DA &> AC + BD \end{aligned}$$

□

---

6. The sum of the lengths of the three medians of a triangle is less than 3/2 times its perimeter.

PROOF. Construct $\triangle ABC$ with medians AF, BE, and CD. We wish to show that

$$AF + BE + CD < \frac{3}{2}(AB + BC + AC)$$

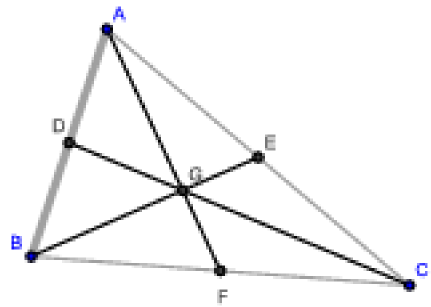


FIGURE 1.0.14. [1.20, #6]

In $\triangle ABE$, notice that by [1.20], we have

$$AE + AB > BE$$

Similarly, in $\triangle DBC$, we have

$$BD + BC > CD$$

and in $\triangle ACF$, we have

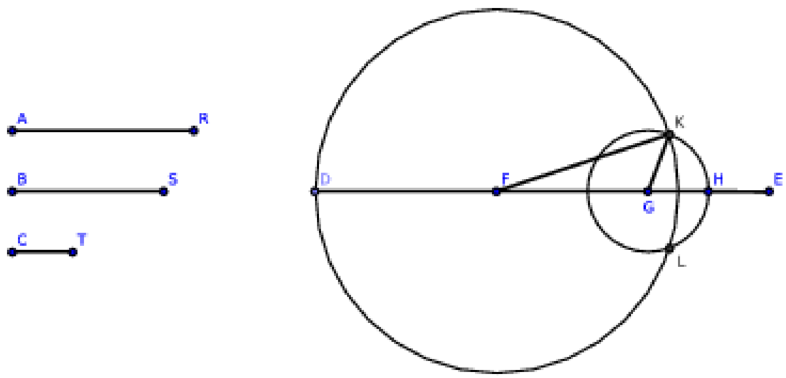
$$AC + CF > AF$$

Adding each inequality, we have that

$$\begin{aligned} AB + BC + AC + AE + BD + CF &> AF + BE + CD \\ AB + BC + AC + \tfrac{1}{2}AC + \tfrac{1}{2}AB + \tfrac{1}{2}BC &> AF + BE + CD \\ \tfrac{3}{2}(AB + BC + AC) &> AF + BE + CD \end{aligned}$$

since $AE = \frac{1}{2}AC$, $BD = \frac{1}{2}AB$, and $CF = \frac{1}{2}BC$. □

[1.22] Exercises



1. Prove that when the above condition is fulfilled (that the sum of every two pairs of segments is greater than the length of the remaining segment) then the two circles must intersect.

PROOF. Construct the figure from [1.22] such that the two circles do not intersect where $AR = FD = FK$, $FG = BS$, and $GH = CT$.

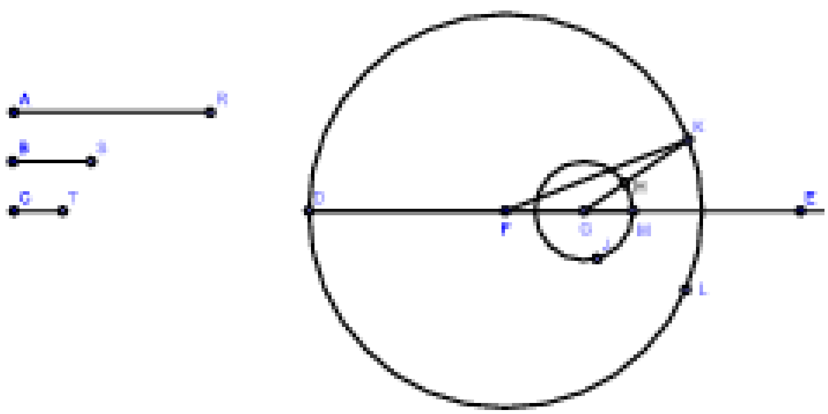


FIGURE 1.0.15. [1.22, #1]

Consider $\triangle FGK$. By [1.20], we have that

$$\begin{array}{rcl} FG + GK & > & FK \\ FG + GH + HK & > & FK \\ FG + GH & > & FK - HK \\ & \Longrightarrow & \\ FG + GH & \leq & FK \\ & \Longrightarrow & \\ BS + CT & \leq & AR \end{array}$$

Hence, when the circles fail to intersect, then that the sum of every two pairs of segments is not greater than the length of the remaining segment; this is the contrapositive statement to that which we set out to prove. □

[1.23] Exercises

1. Construct a triangle given two sides and the angle between them.

PROOF. Suppose we have arbitrary segments AB and CD and an arbitrary angle $\angle EFG$.

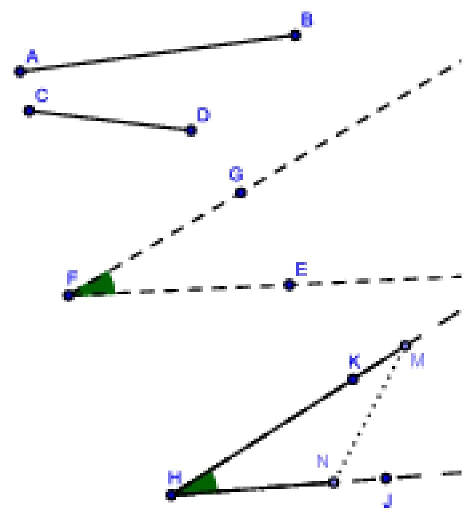
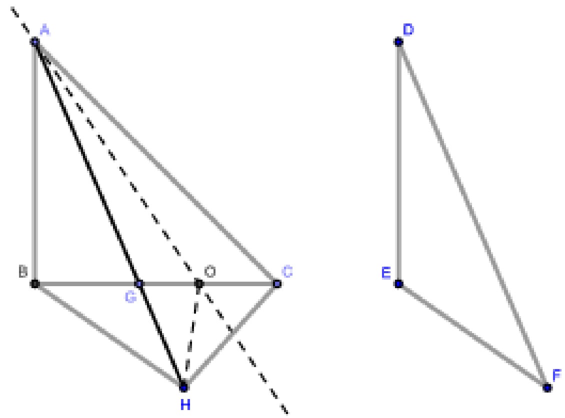


FIGURE 1.0.16. [1.23, #1]

By [1.23], construct rays HJ and HK such that $\angle JHK = \angle EFG$. By [1.4, Cor. 1], we may construct segment HM on ray HJ and segment HN on ray HK such that $AB = HM$ and $CD = HN$. Construct a segment on points M and N [Postulate 1.2]. Then $\triangle MNH$ has sides equal in length to segments AB and CD and contains an angle equal in measure to $\angle EFG$. □

[1.24] Exercises



2. Prove that the angle $\angle BCA > \angle EFD$.

PROOF. By [1.16], $\angle BCA > \angle BHC$, and $\angle BHC = \angle BHA + \angle AHC$. Therefore, $\angle BCA > \angle BHA$. By the proof of [1.24], $\angle BHA = \angle EFD$, and the proof follows. □

[1.26] Exercises

1. The endpoints of the base of an isosceles triangle are equally distant from any point on the perpendicular segment from the vertical angle on the base.

PROOF. By [1.9, #2] and [1.9, #3], any point on the the perpendicular segment from the vertical angle on the base is equally distant from the endpoints of the base. □

2. If the line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

PROOF. Construct $\triangle ABC$ with segment AD such that $\angle BAC = \angle BAD + \angle CAD$ where $\angle BAD = \angle CAD$ and where $BD = DC$. We wish to prove that $\angle CBA = \angle BCA$.

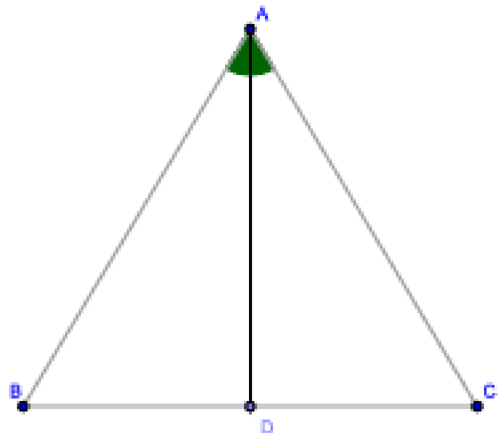


FIGURE 1.0.17. [1.26, #2]

By [1.9, #2], $AD \perp BC$. Consider $\triangle ABD$ and $\triangle ACD$. Notice that $BD = CD$, $\angle BDA = \angle CDA$ (since both are right angles), and $\angle BAD = \angle CAD$. By [1.26], $\triangle ABD \cong \triangle ACD$. Hence, $\angle DBA = \angle DCA$. It follows that

$$\angle CBA = \angle DBA = \angle DCA = \angle BCA$$

□

---

6. Prove that if two right triangles have equal hypotenuses and that if a side of one is equal in length to a side of the other, then the triangles they are congruent. (Note: this proves the special case of Side-Side-Angle congruency for right triangles.)

PROOF. Suppose that $\triangle ABC$ and $\triangle DEF$ are right triangles where $\angle ACB$ and $\angle DFE$ are right angles, the hypotenuses AB and DE are equal in length, and $AC = DF$. (The proof follows *mutatis mutandis* if $BC = EF$.) We wish to show that $\triangle ABC \cong \triangle DEF$.

By superposition, we may "move" $\triangle DEF$ so that side AC lies on side DF such that $A = D$ and $C = F$.

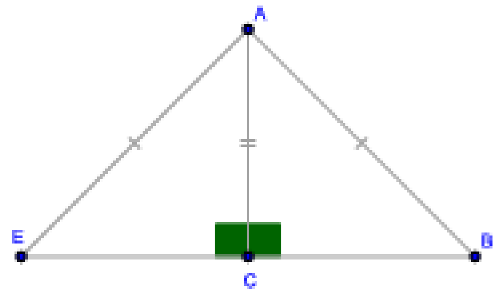


FIGURE 1.0.18. [1.26, #6]

Consider $\triangle AEB$. Since $AE = AB$, $\angle AEB = \angle ABE$ by [1.5].

Now consider $\triangle AEC$ and $\triangle ACB$. Since $\angle ACE = \angle ACB$, $\angle AEC = \angle ABC$, and $AE = AB$, by [1.26], $\triangle AEC \cong \triangle ABC$. Thus $\triangle DEF = \triangle AEC$. □

[1.29] Exercises

2. If $\angle ACD$, $\angle BCD$ are adjacent angles, any parallel to AB will meet the bisectors of these angles at points equally distant from where it meets CD.

PROOF. Construct the straight line AB which contains the point C. Construct a straight line EF such that $AB \parallel EF$ where D is a point on EF. Construct segments CH and CJ such that CJ bisects $\angle ACD$ and CH bisects $\angle BCD$. We wish to show that $DH = DJ$.

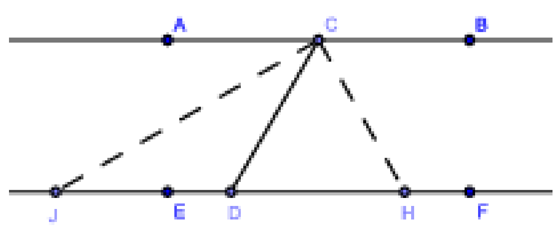


FIGURE 1.0.19. [1.29, #2]

Construct segments JK and HL such that $JK \parallel CD$ and $CD \parallel HL$.

Since CH is a bisector of $\angle BCD$, $\angle BCH = \angle HCD$. Since $CD \parallel HL$, $\angle HCD = \angle CHL$. Similarly, $\angle BCH = \angle DHC$. Hence,

$$\angle BCH = \angle HCD = \angle DHC = \angle CHL$$

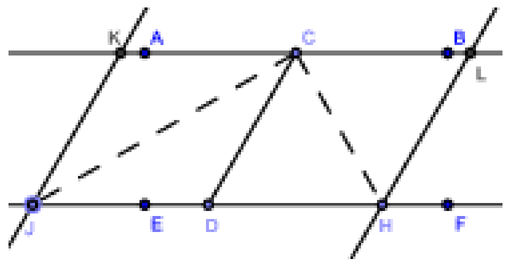


FIGURE 1.0.20. [1.29, #2]

This means that $\triangle CDH$ and $\triangle CLH$ are isosceles triangles where the base angles of each triangle are equal, and so

$$CL = HL = CD = DH$$

By similar reasoning, $\triangle DCJ$ and $\triangle KCJ$ are isosceles triangles where the base angles are equal, and so

$$KJ = CK = CD = DJ$$

Thus, $DJ = DH$. □

---

5. Two straight lines passing through a point equidistant from two parallels intercept equal segments on the parallels.

PROOF. Construct straight lines AB and CD such that $AB \parallel CD$. Construct straight line LM such that $AB \perp LM$ and choose the point G on LM such that $GL = GM$ [1.10]. Construct arbitrary straight lines HJ and IK such that each passes through . We claim that $HI = JK$.

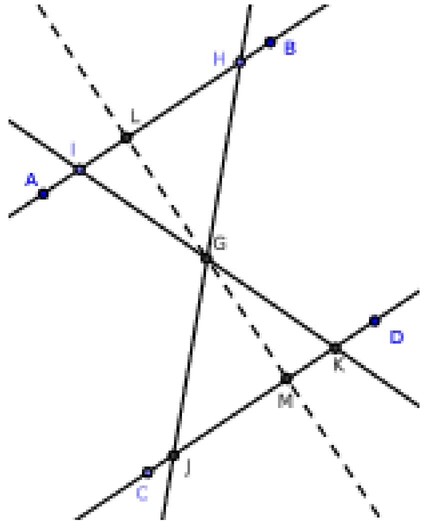


FIGURE 1.0.21. [1.29, #5]

Consider $\triangle GLI$ and $\triangle GMK$: $\angle LGI = \angle MGK$ by [1.15]; $GL = GM$ by construction; $\angle GMK = \angle GLI$ by construction. Hence by [1.26], $\triangle GLI \cong \triangle GMK$. It follows that $GI = GK$.

Now consider $\triangle GHI$ and . By [1.15], $\angle HGI = \angle JGK$. Since $AB \parallel CD$, by [1.29, Cor. 1], $\angle GIH = \angle GKJ$. By the above, $GI = GK$. Again by [1.26], we have that $\triangle GHI \cong \triangle GJK$. Thus, $HI = JK$. □

[1.31] Exercises

1. Given the altitude of a triangle and the base angles, construct the triangle.

PROOF. Suppose we are given altitude h (a segment) and base angles α and β. Extend h to the line AB where A and B are the endpoints of h. Construct the line BC such that $AB \perp BC$ [1.11, Cor. 1].

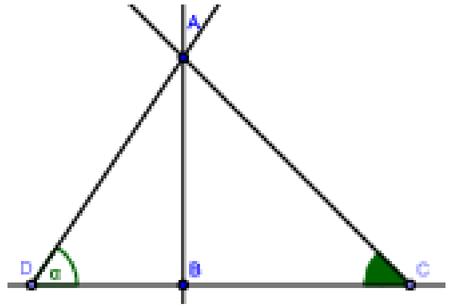


FIGURE 1.0.22. [1.31, #1]

If α is the "left angle" of the triangle to be constructed, then β is the "right angle" of the triangle. (Construction is trivial if the left/right assignment is reversed.) Construct α on the left side of AB as $\angle BDA$ such that the ray at point D intersects point A. Similarly, construct β on the right side of AB as $\angle BCA$ such that the ray at point C intersects point A.

Clearly, figure ACD is a three-sided polygon containing angles α and β whose altitude is h. Hence, $\triangle ACD$ is the required triangle. □

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5. Through two given points on two parallel lines, construct two segments forming a lozenge with given parallels.

PROOF. Construct parallel lines AB and CD where B and C are our given points. We wish to construct the lozenge $FBGC$ by constructing the segments FC and BG.

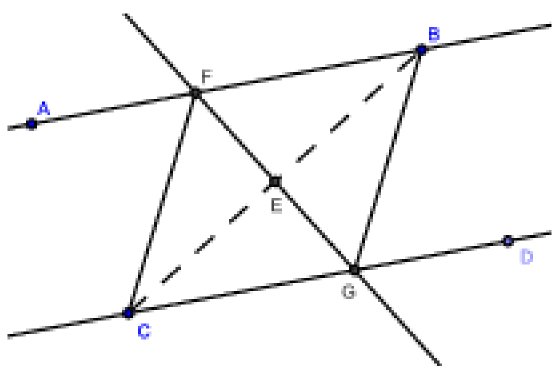


FIGURE 1.0.23. [1.31, #5]

Locate E, the midpoint of BC [1.10]. Construct the line FG such that $FG \perp BC$ and where F is a point on AB and G is a point on CD. Construct segments FC and BG. We claim that $\boxdot FBGC$ is the required lozenge.

By construction, $\boxdot FBGC$ is a parallelogram.

Consider $\triangle CEG$ and $\triangle BFE$. By construction, $BE = CE$ and $\angle CEG = \angle BEF$. By [1.29, Cor. 1], $\angle BFE = \angle CGE$. Then by [1.26], $\triangle CEG \cong \triangle BFE$, and so $BF = CG$ and $EF = EG$.

Now consider $\triangle FEC$ and $\triangle BEG$. We have that $BE = CE$, $EF = EG$, and $\angle FEC = \angle BEG$; by [1.4], $\triangle FEC \cong \triangle BEG$. Hence, $FC = BG$.

By [1.32, #6],

$$\angle FEB = \angle CEG = \angle FEC = \angle GEB = \text{right angle}$$

By [1.47], $BF = BG$, and so

$$BF = BG = FC = CG$$

Thus by [Def 1.29], $FBGC$ is a lozenge. □

[1.32] Exercises.

3. If the line which bisects the external vertical angle is parallel to the base, then the triangle is isosceles.

PROOF. Construct $\triangle ABC$ with external vertical angle $\angle ACE$ where line CD bisects $\angle ACE$ and $CD \parallel AB$. We claim that $\triangle ABC$ is isosceles.

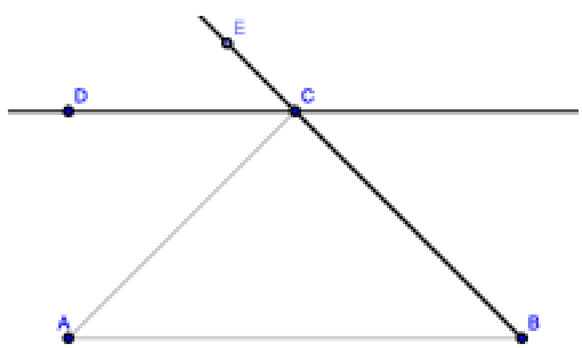


FIGURE 1.0.24. [1.32, #3]

By [1.32], $\angle ACE = \angle CAB + \angle CBA$. By [1.29, Cor. 1], $\angle DCE = \angle CBA$; that the same corollary, $\angle ACD = \angle CAB$. Since $\angle ACD = \angle DCE$ by hypothesis, we have that $\angle ACE = 2 \cdot \angle DCE = 2 \cdot \angle ACD$, from which it follows that $\angle CAB = \angle CBA$. By [1.6], $\triangle ABC$ is isosceles. □

An alternate proof that does not use [1.32]:

PROOF. Suppose the above.

By [1.29, Cor. 1], $\angle DCE = \angle ABC$. By hypothesis, $\angle DCE = \angle ACD$, and so $\angle ACD = \angle ABC$. Again by [1.29, Cor. 1], $\angle ACD = \angle CAB$, and so $\angle CAB = \angle ABC$. By [1.6], $\triangle ABC$ is isosceles. □

---

5. The three perpendicular bisectors of a triangle are concurrent.

PROOF. Construct $\triangle ABC$, altitudes AG and CF, and the segment BE where E is a point on AC such that AG, BE, and CF intersect at point D. We wish to show that $\angle AEB$ is a right angle.

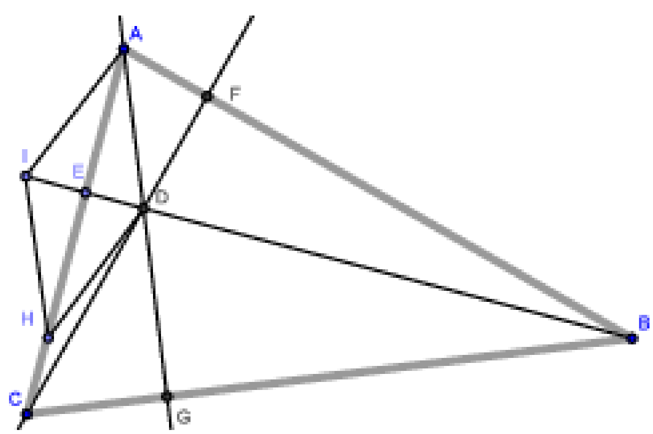


FIGURE 1.0.25. [1.32, #5]

Construct segments HD, HI, and AI such that

$$AD = HD = HI = AI$$

By [Def. 1.29], $\boxdot ADHI$ is a lozenge. By [1.11, #1], the diagonals of a lozenge bisect each other perpendicularly. Since AH is a diagonal of $\boxdot ADHI$, AH is bisected perpendicularly by DI at point E, and so $\angle AED$ equals one right angle. If $\angle EDB$ equals two right angles, then the proof follows.

By the construction of $\angle BDA$, we have that $\angle EDB = \angle BDA + \angle EDA$ and that $\angle BDA = \angle DEA + \angle EAD$ by [1.32]. Hence

$$\angle EDB = \angle BDA + \angle EDA = \angle DEA + \angle EAD + \angle EDA = \text{two right angles}$$

since $\angle DEA$, $\angle EAD$, and $\angle EDA$ are the interior angles of $\triangle EDA$. Thus,

$$\angle AED = \angle AEB = \text{right angle}$$

□

---

6. The bisectors of two adjacent angles of a parallelogram are at right angles.

PROOF. Construct a parallelogram. If the parallelogram is a lozenge, the result follows from [1.34].

Otherwise, construct parallelogram $\boxdot ABCD$ which is not a lozenge. Construct $\angle DAF$ such that $\angle DAF = \frac{1}{2}\angle DAB$ and $\angle ABE$ such that $\angle ABE = \frac{1}{2}\angle ABC$. Extend AD to meet BE at G and extend BC to meet AF at H. Connect G and H. Since $BC \parallel AD$, $BH \parallel AG$; similarly, $BA \parallel HG$. Hence, $\boxdot ABHG$ is a parallelogram. We claim that $\boxdot ABHG$ is also a lozenge.

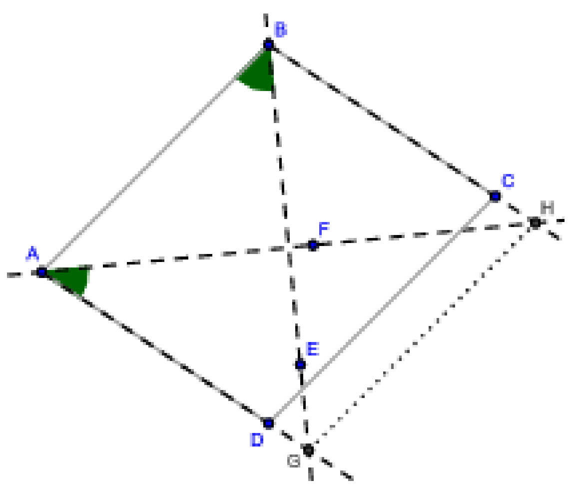


FIGURE 1.0.26. [1.32, #6]

By hypothesis, $\angle GAH = \angle BAH$. By [1.29, Cor. 1], $\angle GAH = \angle AHB$ and $\angle BAH = \angle AHG$. Hence

$$\angle GAH = \angle BAH = \angle AHB = \angle AHG$$

Consider $\triangle ABH$ and $\triangle AGH$. Since $\angle BAH = \angle AHB$, $\triangle ABH$ is isosceles. A similar argument holds for $\triangle AGH$. Given the equality of the four base angles above and given than the two triangles share base AH, by [1.26] $\triangle ABH \cong \triangle AGH$. Hence, we have that

$$AB = BH = HG = GA$$

By [Def 1.29], $\boxdot ABHG$ is a lozenge. □

[1.33] Exercises

1. Prove that if two segments AB, BC are respectively equal and parallel to two other segments DE, EF, then the segment AC joining the endpoints of the former pair is equal in length to the segment DF joining the endpoints of the latter pair.

PROOF. Construct segments AB, BC, DE, EF such that $AB = DE$, $BC = EF$, $AB \parallel DE$, and $BC \parallel EF$. Construct segments AC and DF. We wish to show $AC = DF$.

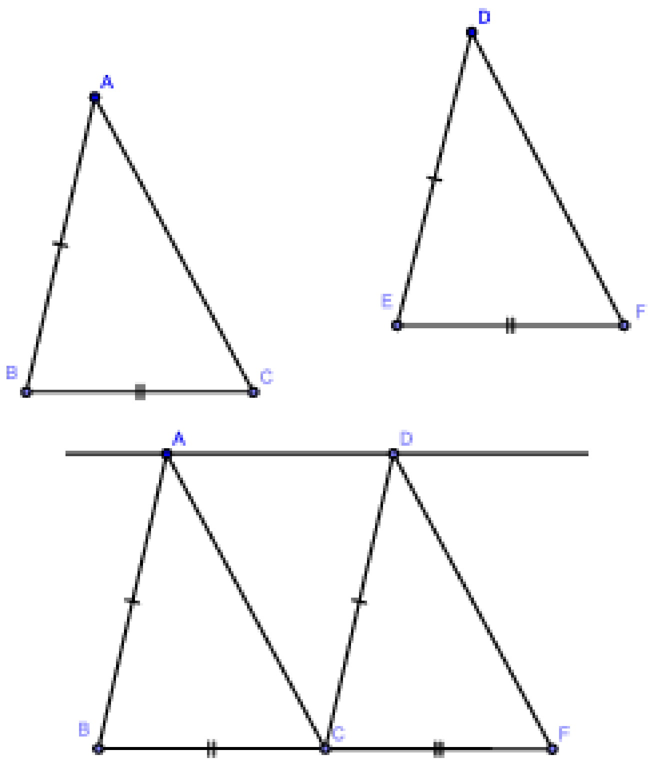


FIGURE 1.0.27. [1.33, #1]

Consider $\triangle ABC$ and $\triangle DEF$. Suppose that $\angle ABC \neq \angle DEF$. Move $\triangle ABC$ such that $C = E$. Construct AD such that $AD \parallel BF$. Since $AB \parallel CD$ and $AD \parallel BF$, we have that $\angle ABC = \angle DCF$. But $\angle DCF = \angle DEF \neq \angle ABC$, a contradiction. Hence, $\angle ABC = \angle DEF$. By [1.4], $\triangle ABC \cong \triangle DEF$, and so $AC = DF$. □

[1.34] Exercises

1. Show that the diagonals of a parallelogram bisect each other.

PROOF. Consider $\boxdot ABCD$ and diagonals AD, BC. Let point E be the intersection of AD, BC. We wish to prove that $CE = EB$ and $AE = ED$.

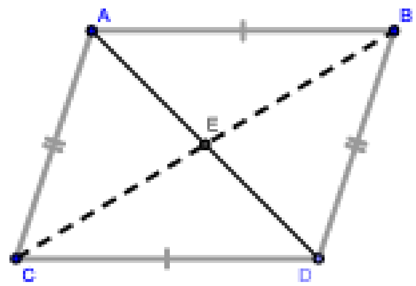


FIGURE 1.0.28. [1.34], #1

Since $AB \parallel CD$, $\angle BCD = \angle CBA$. Similarly, we have that $\angle CDA = \angle DAB$. Consider $\triangle ECD$ and $\triangle AEB$. Since $\angle ECD = \angle EBA$, $\angle EDC = \angle EAB$, and $CD = AB$, by [1.26] we have that $\triangle ECD \cong \triangle AEB$. Hence, $AE = ED$. A similar argument shows that $CE = EB$, *mutatis mutandis*. Since both diagonals are bisected at their point of intersection, the proof follows. □

---

2. If the diagonals of a parallelogram are equal, each of its angles are right angles.

PROOF. Construct $\boxdot ABCD$ as in [1.34, #1] and suppose that $AD = BC$. We wish to show that

$$\angle CAB = \angle ABD = \angle BDC = \angle ACD = 1 \text{ right angle}$$

By [1.34, #1], we have that $EA = ED = EC = EB$. Consider $\triangle ECD$, $\triangle EDB$, $\triangle EBA$, and $\triangle EAC$; by the above, they are isosceles triangles. By [1.6, Cor. 1],

$$\angle EAC = \angle ECA = \angle ECD = \angle EDC = \angle EDB = \angle EBD = \angle EBA = \angle EAB$$

Consider $\triangle ABC$. Notice that

$$\begin{aligned} \angle ECA + \angle EAC + \angle EAB + \angle EBA &= 2 \text{ right angles} \\ 4 \cdot \angle ECA &= 2 \text{ right angles} \\ \angle ECA &= \tfrac{1}{2} \text{ right angles} \end{aligned}$$

By the above equality, it follows that

$$\angle CAB = \angle ABD = \angle BDC = \angle ACD = 1 \text{ right angle}$$

□

6. Construct a triangle being given the midpoints of its three sides.

PROOF. Suppose we are given midpoints A, B, and C of a triangle. We wish to construct $\triangle DEF$ such that $DA = AE$, $DC = CF$, and $EB = BF$.

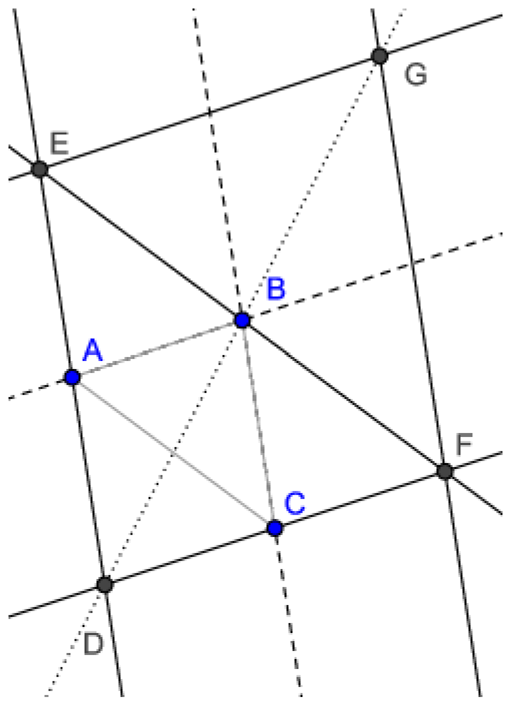


FIGURE 1.0.29. [1.34, #6]

Construct $\triangle ABC$. Also construct line AD such that $AD \parallel BC$, construct line BE such that $BE \parallel AC$, extend BC to a line, extend AB to a line, construct line CD such that $CD \parallel AB$, and construct line BD. We claim that $\triangle DEF$ is the required triangle.

Notice that if we construct segments EG and FG such that $\boxdot EGFD$ is a parallelogram, the point B intersects the two diagonals of $\boxdot EGFD$. By [1.34, #1], the diagonals are bisected. Hence, $EB = BF$. By similar constructions, we may show that $DA = AE$ and $DC = CF$, *mutatis mutandis*. (The details are left as an exercise to the student.) $\square$

[1.37] Exercises

1. If two triangles of equal area stand on the same base but on opposite sides, the segment joining their vertices is bisected by the base.

PROOF. Suppose we have $\triangle ABG$ and $\triangle ABI$ such that both triangles share base AB and point G stands on the opposite side of AB than point I. We claim that the segment GI is bisected by either AB or by the extension of AB to a line.

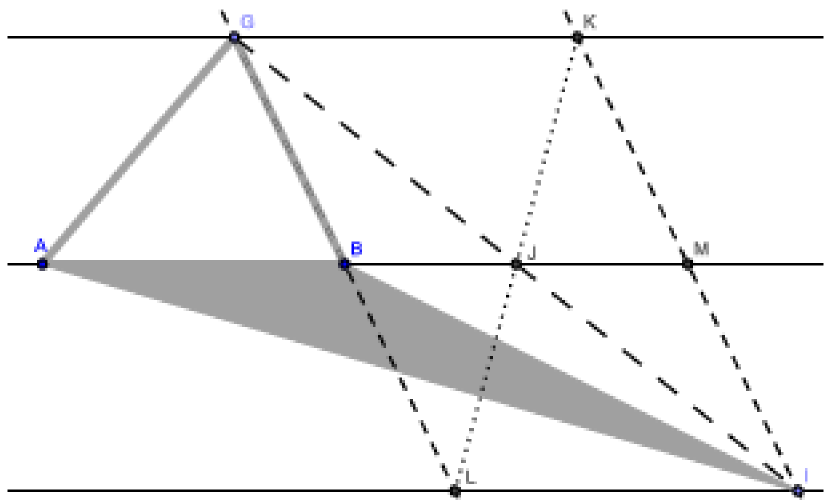


FIGURE 1.0.30. [1.37, #1]

Extend AB to a line if necessary. Let J be the point where GI intersects AB or its extension. Construct lines GK and LI such that $GK \parallel AB$ and $LI \parallel AB$. By [1.30], $GK \parallel LI$. Also construct lines GB and KI such that $GB \parallel KI$. Hence, $\boxdot GKIL$ is a parallelogram. By [1.34, #1], $GJ = JI$. Hence, the proof. □

[1.38] Exercises

1. Every median of a triangle bisects the triangle.

PROOF. Construct $\triangle ABC$ where AF is the median of side BC. We wish to show that $\triangle ABF = \triangle ACF$.

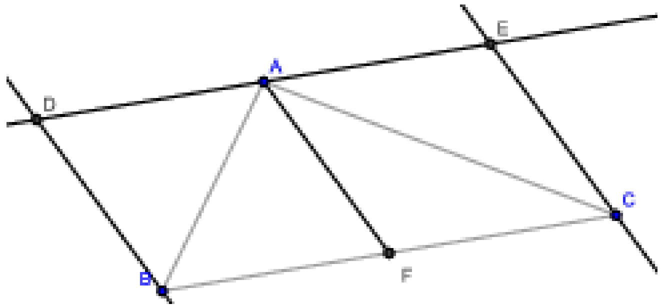


FIGURE 1.0.31. [1.38, #1]

Construct line AE such that $AE \parallel BC$. Clearly, $\triangle ABF$ and $\triangle ACF$ stand between the same parallels (AE and BC). Since $BF = FC$ by hypothesis, $\triangle ABF = \triangle ACF$ by [1.38]. □

Note: we do **not** claim that the triangles are congruent, merely equal in area.

5.One diagonal of a quadrilateral bisects the other if and only if it also bisects the quadrilateral.

PROOF. Suppose we have quadrilateral $ABCD$ with diagonal BD such that BD intersects AC at E such that $AE = EC$. We claim that $\triangle ABD = \triangle CBD$.

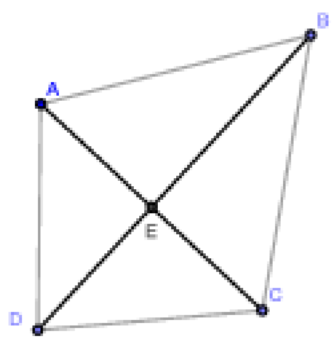


FIGURE 1.0.32. [1.38, #5]

Consider $\triangle ABE$ and $\triangle CBE$. By [1.38], they are equal in area; a similar result holds for $\triangle AED$ and $\triangle CED$.

Now consider $\triangle ADB$ and $\triangle CDB$. Notice that

$$\triangle ADB = \triangle AED \oplus \triangle ABE = \triangle CED \oplus \triangle CBE = \triangle CDB$$

Now suppose that $\triangle ABD = \triangle CBD$. By [1.37, #1], we have that $AE = EC$. □

[1.40] Exercises

1. Triangles with equal bases and altitudes are equal in area.

PROOF. Suppose we have two triangles with equal bases and with equal altitudes. Since the altitude of a triangle is the distance between the parallels which contain it, equal altitudes imply that the triangles stand between the same parallels. By [1.38], the triangles are equal in area. □

---

2. The segment joining the midpoints of two sides of a triangle is parallel to the third because the medians from the endpoints of the base to these points will each bisect the original triangle. Hence, the two triangles whose base is the third side and whose vertices are the points of bisection are equal in area.

PROOF. Construct $\triangle ABC$ with midpoint D on side AB and midpoint E on side AC. Connect D and E. We wish to show that $DE \parallel BC$.

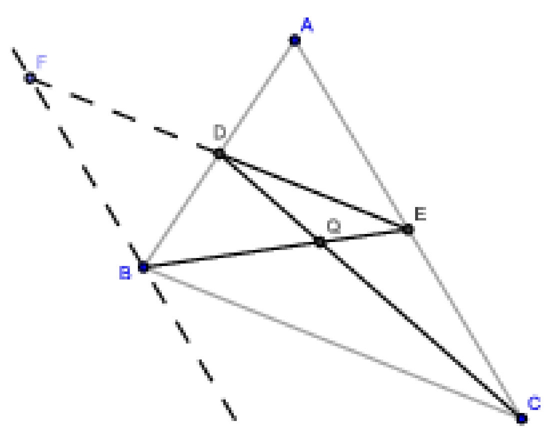


FIGURE 1.0.33. [1.40, #2]

Construct segment DF such that $DE = DF$. Construct line BF and consider $\triangle ADE$ and $\triangle BDF$. Notice that $\angle ADE = \angle BDE$ by [1.15]. Also, since D bisects AB, $AD = BD$. Finally, $DE = DF$ by construction. Hence by [1.4], $\triangle ADE \cong \triangle BDF$, and so $\angle FBD = \angle DAE$. By [1.29, Cor. 1], $FB \parallel AC$.

Also, since $FB = AE$ and $AE = EC$, $FB = EC$. Since FB and EC are equal in length and parallel, by [1.33] EF and BC are opposite and parallel; hence, $\boxdot FECB$ is a parallelogram. Thus, $DE \parallel BC$. □

COROLLARY. *$BC = 2 \cdot DE$, or the median segment as constructed is half the length of its opposite and parallel side.*

---

4. The segments which connect the midpoints of the sides of a triangle divide it into four congruent triangles.

PROOF. Suppose we have $\triangle ABC$ with midpoints D on side BC, E on side AC, and F on side AB. Construct segments DE, EF, and DF. We wish to show that

$$\triangle AEF \cong \triangle ECD \cong \triangle FDB \cong \triangle DFE$$

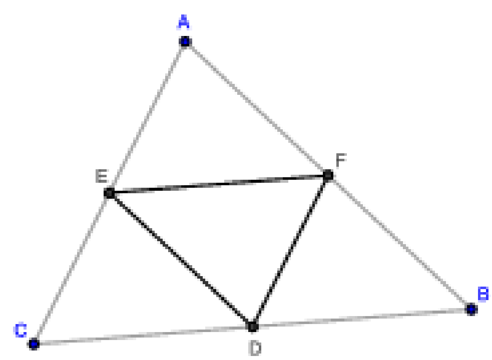


FIGURE 1.0.34. [1.40, #4]

By [1.40, #2], we have that $DE \parallel AB$, $DF \parallel AC$, and $EF \parallel BC$. It follows that by [1.29, Cor. 1], we have that

$$\angle EDC = \angle DEF = \angle EFA = \angle DBF$$

Similarly by [1.29, Cor. 1], we have that $\angle CED = \angle EAF = \angle FDB = \angle DFB$. Since $\boxdot AFDE$ is a parallelogram, we have that $\angle EAF = \angle EDF$ [1.34], or

$$\angle CED = \angle EAF = \angle EDF = \angle DFB$$

Since $\boxdot EFCD$ is a parallelogram, we have that $EF = CD$. Since $\boxdot EFBD$ is a parallelogram, we have that $EF = BD$, or

$$EF = CD = BD$$

By [1.26],

$$\triangle AEF \cong \triangle DEF \cong \triangle ECD \cong DBF$$

□

[1.46] Exercises

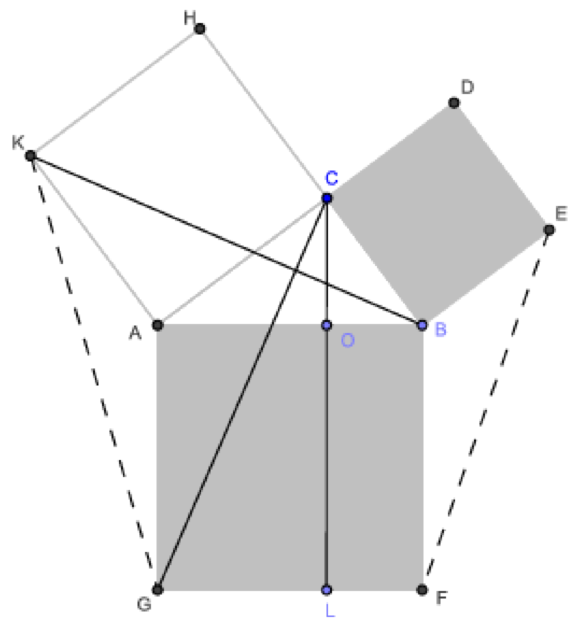
1. Two squares have equal side-lengths if and only if the squares are equal in area.

PROOF. Suppose we have two squares with equal side-length. Divide each square into two equal triangles by constructing a diagonal. The side-length is then the altitude of each triangle. By [1.34], each triangle is half of the area of the square. By [1.40, #1], each triangle is equal in area to its corresponding triangle in the other square. Together, these results show that all four triangles are equal in area, and therefore the squares are equal in area.

Now suppose that we have two squares which are equal in area where the side-lengths are unequal. Divide each square into triangles by the above method. It follows that each side of a triangle from the larger square has a longer altitude than its corresponding side on the smaller triangle. It follows that the areas of the triangle are unequal.

Thus, the proof. □

[1.47] Exercises



10. Each of the triangles $\triangle AGK$ and $\triangle BEF$ formed by joining adjacent corners of the squares is equal in area to the right triangle $\triangle ABC$.

PROOF. Construct the polygons as in [1.47] and then construct segments AG and EF. We wish to show that

$$\triangle ABC = \triangle AGK = \triangle BEF$$

We will employ trigonometry to solve this exercise. First, we note that one right angle equals $\frac{\pi}{2}$ radians.

Consider $\triangle ABC$. Notice that if $\angle CAB = \gamma$, then $\angle ABC = \frac{\pi}{2} - \gamma$ since $\angle ACB = \frac{\pi}{2}$. It follows that

$$\angle KAG = 2\pi - (\frac{\pi}{2} + \frac{\pi}{2} + \angle CAB) = \pi + \gamma$$

and that

$$\angle FBE = 2\pi - (\frac{\pi}{2} + \frac{\pi}{2} + \angle ABC) = \frac{\pi}{2} - \gamma$$

We will employ the general form of the equation of the area of a triangle:

$$\text{Area} = \frac{1}{2}xy \cdot \sin\theta$$

where θ is the interior angle to sides x and y.

$$\begin{aligned} \text{Area } \triangle AKG &= \tfrac{1}{2}AK \cdot AG \cdot \text{sin}(\pi + \gamma) \\ &= \tfrac{1}{2}AK \cdot AG \cdot \text{sin}(\gamma) \end{aligned}$$

by the properties of the sin function. We also have that

$$\begin{aligned}\text{Area } \triangle ABC &= \tfrac{1}{2}AB \cdot AC \cdot \sin(\gamma) \\ &= \tfrac{1}{2}AK \cdot AG \cdot \sin(\gamma)\end{aligned}$$

and so

$$\triangle AKG = \triangle ABC$$

Similarly,

$$\begin{aligned}\text{Area } \triangle BEF &= \tfrac{1}{2}BE \cdot BF \cdot \sin(\pi + \gamma) \\ &= \tfrac{1}{2}BC \cdot AB \cdot \cos(\gamma)\end{aligned}$$

and

$$\begin{aligned}\text{Area } \triangle ABC &= \tfrac{1}{2}AB \cdot BC \cdot \sin(\tfrac{\pi}{2} - \gamma) \\ &= \tfrac{1}{2}AB \cdot BC \cdot cos(\gamma)\end{aligned}$$

by the properties of the cosine function.

Thus, the proof. □

Chapter 1 exercises

1. Any triangle is equal to a fourth part of the area which is formed by constructing through each vertex a line which is parallel to its opposite side.

PROOF. Construct $\triangle ABC$; then construct $DF \parallel BC$ through point A, construct $EF \parallel AB$ through point C, and construct $DE \parallel AC$ through point B. We wish to show that the area of $\triangle DEF$ is equal to four times the area of $\triangle ABC$.

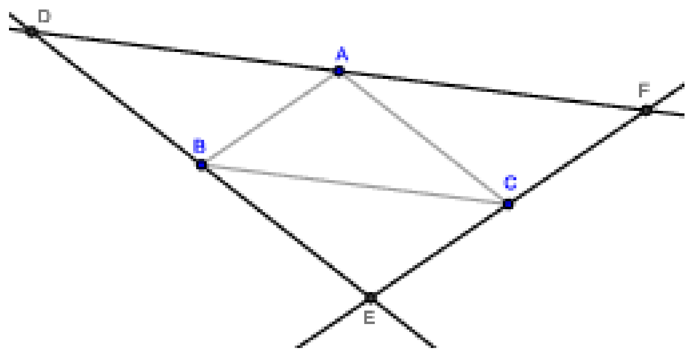


FIGURE 1.0.35. Chapter 1 exercises, #1

Since $DF \parallel BC$, $\angle ADB = \angle CBE$. Since $DE \parallel AC$, $\angle ADB = \angle CAF$. Hence,

$$\angle ADB = \angle CBE = \angle CAF$$

Similarly, we can show that

$$\angle CFA = \angle BAD = \angle ECB$$

By [1.32], it follows that

$$\angle DBA = \angle BEC = \angle ACF$$

Notice that $\boxdot AFCB$ is a parallelogram. Hence, $AF = BC$. Since $\boxdot DACB$ is also a parallelogram, $BC = DA$. By [1.26],

$$\triangle ABD \cong \triangle CEB \cong \triangle FCA$$

By [1.8],

$$\triangle ABD \cong \triangle CEB \cong \triangle FCA \cong \triangle ABC$$

Since $\triangle DEF = \triangle ABD \oplus \triangle CEB \oplus \triangle FCA \oplus \triangle ABC$, the proof follows. □

---

4. The three medians of a triangle are concurrent. (Note: we are proving the existence of the **centroid** of a triangle.)

PROOF. Construct $\triangle ABC$ where CE is the median of AB and BF is the median of AC. Through the intersection of CE and BF, point G, construct the segment AGD where D is a point on BC. We wish to show that $BD = DC$.

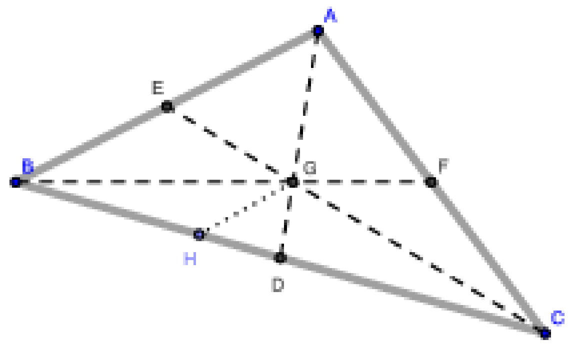


FIGURE 1.0.36. Chapter 1 exercises, #4

By [1.38, #1], notice that

$$\triangle BGE \oplus \triangle EGA \oplus \triangle AGF = \triangle FGC \oplus \triangle CGD \oplus \triangle DGB$$

and

$$\triangle CGD \oplus \triangle DGB \oplus \triangle BGE = \triangle EGA \oplus \triangle AGF \oplus \triangle FGC$$

It follows that

$$\begin{aligned}
\triangle BGE \oplus \triangle EGA \oplus \triangle AGF \ominus \triangle FGC &= \triangle CGD \oplus \triangle DGB \\
2 \cdot \triangle BGE \oplus \triangle EGA \oplus \triangle AGF \ominus \triangle FGC &= \triangle CGD \oplus \triangle DGB \oplus \triangle BGE \\
2 \cdot \triangle BGE \oplus \triangle EGA \oplus \triangle AGF \ominus \triangle FGC &= \triangle EGA \oplus \triangle AGF \oplus \triangle FGC \\
2 \cdot \triangle BGE \ominus \triangle FGC &= \triangle FGC \\
2 \cdot \triangle BGE &= 2 \cdot \triangle FGC \\
\triangle BGE &= \triangle FGC
\end{aligned}$$

By [1.38], we have that $\triangle BGE = \triangle EGA$ and $\triangle FGC = \triangle FGA$, and so

$$\triangle BGE = \triangle EGA = \triangle FGC = \triangle FGA$$

Suppose H, a point on BD other than B or D, is the midpoint of BC. Then $BH = HC$ and by [1.38], $\triangle BGH = \triangle CGH$, or

$$\begin{aligned}
\triangle BGH &= \triangle CGD \oplus \triangle HGD \\
\triangle BGH \ominus \triangle HGD &= \triangle CGD
\end{aligned}$$

Also notice that by [1.38, #1],

$$\begin{aligned}
\triangle ABD &= \triangle ACD \\
\triangle EGA \oplus \triangle BGE \oplus \triangle BGH \oplus \triangle HGD &= \triangle FGA \oplus \triangle FGC \oplus \triangle CGD \\
\triangle BGH \oplus \triangle HGD &= \triangle CGD
\end{aligned}$$

By the above, we have that

$$\triangle BGH \ominus \triangle HGD = \triangle BGH \oplus \triangle HGD$$

which can only hold if $\triangle HGD$ has no area, a contradiction. A similar result follows if H is a point on CD which is neither C nor D.

Thus, $BD = DC$. □

COROLLARY. *The three medians of a triangle divide the original triangle into six sub-triangles which are each equal in area.*

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8. Construct a triangle given the three medians.

PROOF. Suppose we are given the medians of a triangle AE, BF, and CD. We wish to construct the triangle to which they belong.

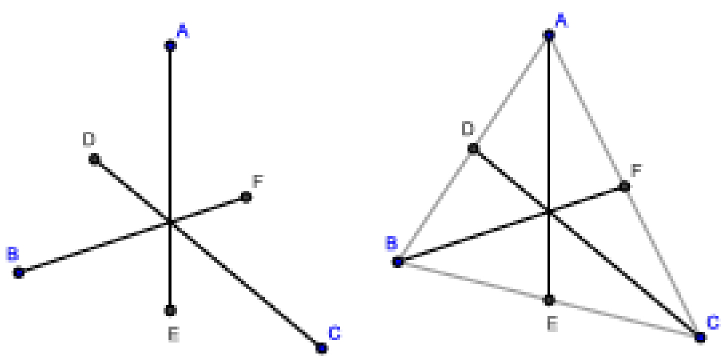

FIGURE 1.0.37. Chapter 1 exercise #8

Connect points A and B; connect A and C; finally, connect B and C. Since AE is a median, BEC forms a segment where $BE = EC$. Similar statements can be made for the remaining side, *mutatis mutandis*. Hence, the required triangle has been constructed. □

---

12. The shortest median of a triangle corresponds to the largest side.

PROOF. Construct $\triangle ABC$ where $BC > AC > AB$, CE is the median of AB, BF is the median of AC, and AD is the median of BC. We wish to show that $AD < FB$ and $AD < EC$.

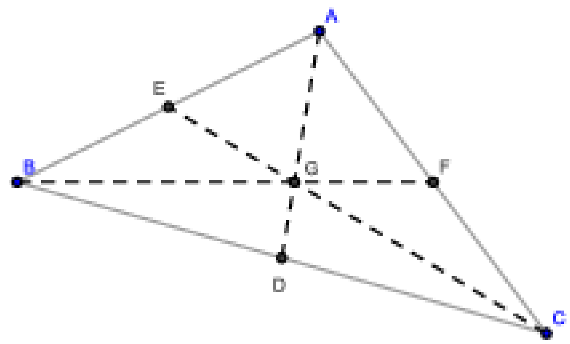

FIGURE 1.0.38. Chapter 1 exercises, #12

Consider $\triangle BGD$ and $\triangle BGE$. By the Corollary to [Ch.1 Exercises, #4], we have that $\triangle BGD = \triangle BGE$. Since $BD > BE$ (because $BC > AB$, $BD = \frac{1}{2}BC$. and $BE = \frac{1}{2}BA$) , and since the triangles share side BG, we must have that $EG > DG$. Similarly, we can show that $EG > FG > DG$. By [Ch. 6 Exercises. #124], we have that the medians of a triangle divide each other in the ratio of $2 : 1$. Hence, $EC > FB > AD$. □

---

16. Inscribe a lozenge in a triangle having for an angle one angle of the triangle.

PROOF. Construct $\triangle ABC$. Let $\angle DAB$ bisect $\angle CAB$ where D is a point on the side BC. Construct the ray AD. Construct $\angle ADF$ and $\angle ADE$ such that

$$\angle ADF = \angle DAB = \angle ADE$$

We claim that $\boxdot AEDF$ is the required lozenge.

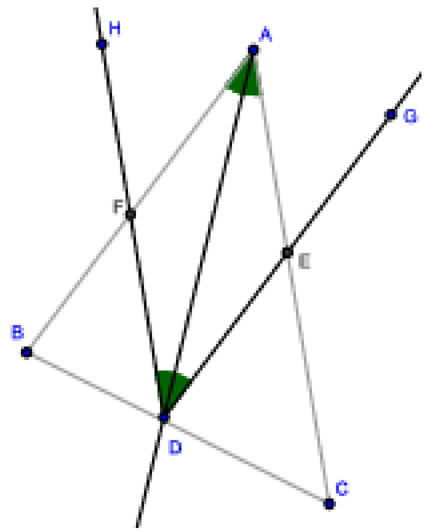

FIGURE 1.0.39. [Ch. 1 Exercises, #16]

Consider $\triangle DFA$. Since it is an isosceles triangle, $DF = FA$ [1.6]. Also consider $\triangle DEA$. Notice that by a similar argument, we have that $AE = ED$. Since $\triangle DFA$ and $\triangle DEA$ share side AD, by [1.26] we have that $\triangle DFA \cong \triangle DEA$. Hence,

$$DF = FA = AE = ED$$

By [Def. 1.29], $\boxdot AEDF$ is a lozenge. Since $\triangle ABC$ and $\boxdot AEDF$ share $\angle BAC$, the proof follows. $\square$

CHAPTER 2

Solutions: Rectangles

[2.4] Exercises

2. If from the vertical angle of a right triangle a perpendicular falls on the hypotenuse, its square equals the area of the rectangle contained by the segments of the hypotenuse.

PROOF. Construct right triangle $\triangle ABC$ where $\angle BAC$ is the vertical angle. Construct segment AD such that $AD \perp BC$. We claim that $AD^2 = DB \cdot DC$.

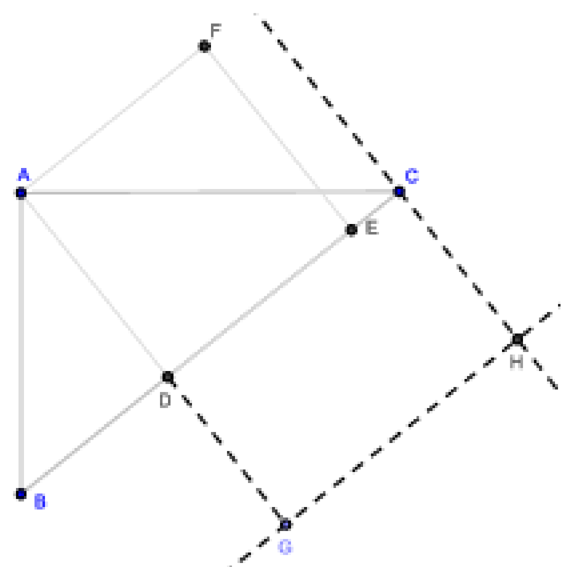

FIGURE 2.0.1. [2.4, #2]

Construct rectangle $\boxdot DCHG$ where $BD = DG$. Geometrically, we claim that $AD^2 = \boxdot DCHG$.

By [1.47], we have that

$$\begin{array}{rcl} AD^2 + DC^2 & = & AC^2 \\ AD^2 + DB^2 & = & AB^2 \end{array}$$

as well as

$$\begin{aligned} AB^2 + AC^2 &= (DB + DC)^2 \\ AB^2 + AC^2 &= DB^2 + 2 \cdot DB \cdot DC + DC^2 \end{aligned}$$

Hence,

$$\begin{aligned} AD^2 + DC^2 + AD^2 + DB^2 &= DB^2 + 2 \cdot DB \cdot DC + DC^2 \\ 2 \cdot AD^2 &= 2 \cdot DB \cdot DC \\ AD^2 &= DB \cdot DC \end{aligned}$$

□

[2.6] Exercises

7. Give a common statement which will include [2.5] and [2.6].

PROOF. Let AB be a line and consider the segment formed by points A and B. Locate the midpoint C of this segment. Choose a point D on the line AB such that D is neither A, B, nor C. We then have two cases:

1) D is between A and B. By [2.5], we have that

$$AD.DB + CD^2 = CB^2$$

2) D is not between A and B. By [2.6], we have that

$$AD.DB + CB^2 = CD^2$$

□

[2.11] Exercises

3. If AB is cut in "extreme and mean ratio" at C, prove that

(a) $AB^2 + BC^2 = 3AC^2$

PROOF. By [2.11], $x = -\frac{a}{2}(1 \pm \sqrt{5})$. (We may ignore negative results since our context is the side-length of planar figures.) Since $AB = a$, $BC = a - x$, and $AC = x$, we have:

$$\begin{aligned} AB^2 + BC^2 &= 3AC^2 \\ a^2 + (a-x)^2 &= 3x^2 \\ a^2 + a^2 - 2ax + x^2 &= 3x^2 \\ 2a^2 - 2ax &= 2x^2 \\ a^2 - ax &= x^2 \\ a^2 + \frac{a^2}{2}(1+\sqrt{5}) &= \frac{a^2}{2}(3+\sqrt{5}) \\ \frac{a^2}{2}(3+\sqrt{5}) &= \frac{a^2}{2}(3+\sqrt{5}) \end{aligned}$$

□

Chapter 2 exercises

15. Any rectangle is equal in area to half the rectangle contained by the diagonals of squares constructed on its adjacent sides.

PROOF. Construct rectangle $ADCB$, squares $GABH$ and $BCFE$, and diagonals GB and BF. We claim that $\frac{1}{2}GB^2 \cdot BF^2 = AB \cdot BC$.

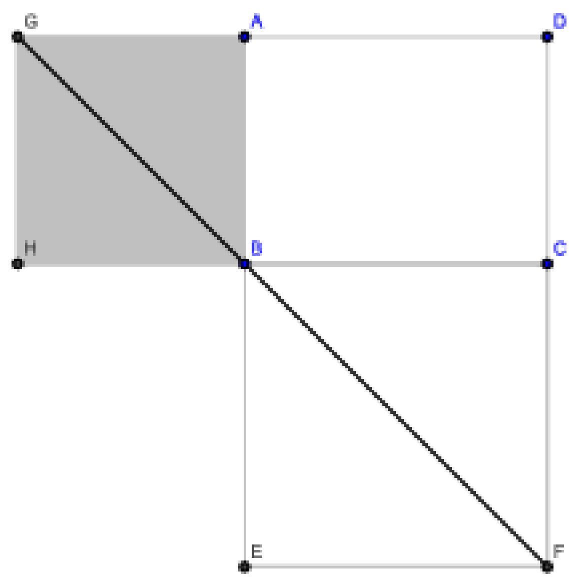

FIGURE 2.0.2. [Ch. 2 Exercises, #15]

Let $AB = x$ and $BC = y$. By [1.47], it follows that $GB = x\sqrt{2}$ and $BC = y\sqrt{2}$. Notice that

$$\begin{aligned}\tfrac{1}{2}GB^2 \cdot BF^2 &= \tfrac{1}{2}xy\sqrt{4}\\ &= xy\\ &= AB \cdot BC\end{aligned}$$

□

CHAPTER 3

Solutions: Circles

[3.3] Exercises

3. Prove [3.3, Cor. 4]: The line joining the centers of two intersecting circles bisects their common chord perpendicularly.

PROOF. Construct the circles from [1.1], and then construct segment CF. We wish to show that AB bisects CF and that $AB \perp CF$.

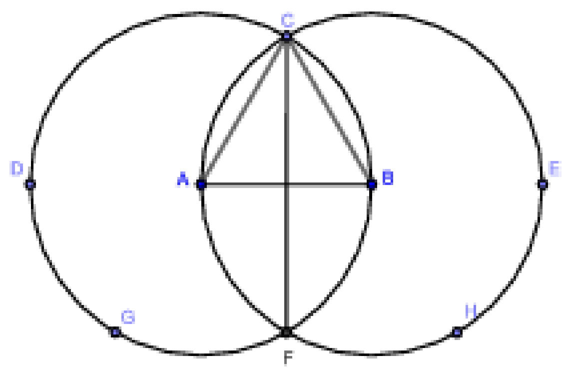

FIGURE 3.0.1. [3.3, #3]

The proof follows immediately from the proof of [1.1, #2]. The details are left as an exercise to the reader. □

[3.5] Exercises

2. Two circles cannot have three points in common without coinciding.

PROOF. Suppose that two circles ($\circ EDF$, $\circ EBF$) have three points in common (E, F, and G) and do not coincide. By [3.3, Cor. 4], the line joining the centers of two intersecting circles (AC) bisects their common chord perpendicularly. Hence, AC bisects EF; similarly, AC bisects EG. But EG can be constructed to that EG and AC do not intersect, a contradiction.

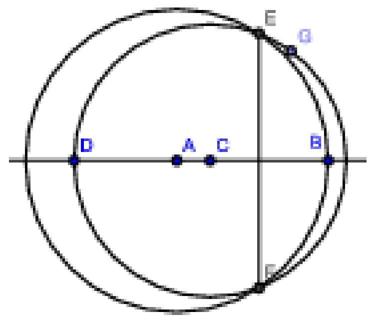

FIGURE 3.0.2. [3.5, #2]

Hence, $\circ EDF$ and $\circ EBF$ coincide. □

[3.13] Exercises

3. Suppose two circles touch externally. If through the point of intersection any secant is constructed cutting the circles again at two points, the radii constructed to these points are parallel.

PROOF. Suppose $\circ GBD$ and $\circ EFB$ touch at point B. By [3.13], these circles touch only at B. Construct secant DBE. We claim that $AD \parallel CE$.

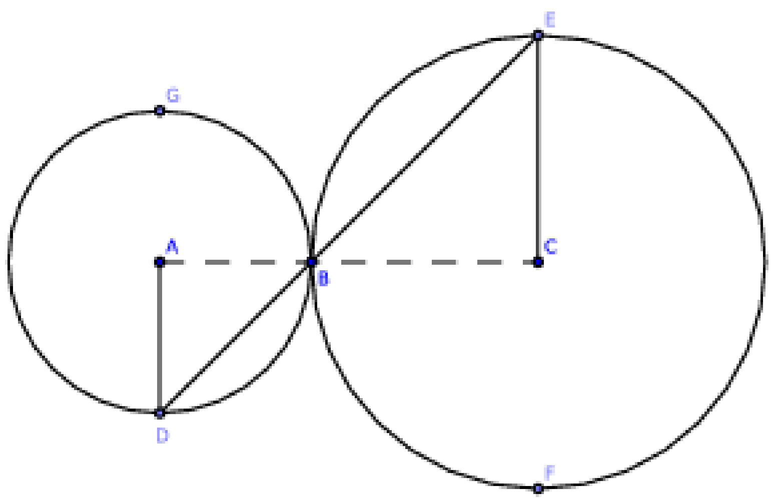

FIGURE 3.0.3. [3.13, #3]

Connect AC. By [3.12], AC intersects B.

Consider $\triangle ABD$ and $\triangle CBE$. Notice that $\angle ABD = \angle CBE$ by [1.15]. Also, since each triangle is isosceles, $\angle ADB = \angle ABD$ and $\angle CEB = \angle CBE$. Hence,

$$\angle ADB = \angle CEB$$

By [1.29, Cor. 1], $AD \parallel CE$. □

COROLLARY. *1. If two circles touch externally and through the point of intersection any secant is constructed cutting the circles again at two points, the diameters constructed to these points are parallel.*

---

4. Suppose two circles touch externally. If two diameters in these circles are parallel, the line from the point of intersection to the endpoint of one diameter passes through the endpoint of the other.

PROOF. Suppose $\circ DBE$ and $\circ FBG$ touch at point B. By [3.13], these circles touch only at B. Construct diameters DE and FG such that $DE \parallel FG$. We claim that the line DB intersects G. (The case for the remaining endpoints follows *mutatis mutandis*.)

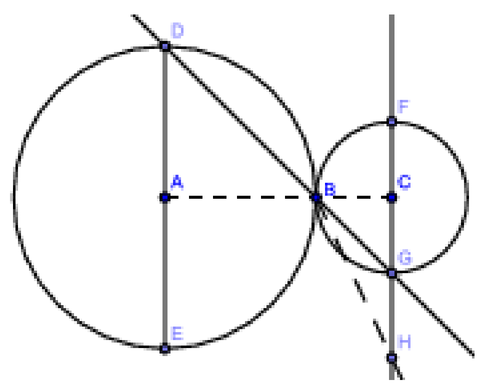

FIGURE 3.0.4. [3.13, #4]

Suppose that DB does not intersect G. Extend FG to a line, and suppose that DB intersects the line FG at H.

Construct the segment AC. By [1.15], $\triangle ABD = \triangle CBH$. By [1.29, Cor. 1], $\angle ADB = \angle CHB$. It follows that $\triangle ABD$ and $\triangle CBH$ are equiangular. However, $\triangle ABD$ is an isosceles triangle and $\triangle CBH$ is not since $CH > CG$; hence, the triangles are not equiangular, a contradiction. A similar contradiction is obtained if DB intersects the line FG at any point other than G, *mutatis mutandis*.

The proof follows. □

[3.16] Exercises

1. If two circles are concentric, all chords of the greater circle which touch the lesser circle are equal in length.

PROOF. Construct $\circ BFG$ and $\circ DCE$ each with center A. On $\circ DCE$, construct chord DE such that DE touches $\circ BFG$ at B. Also on $\circ DCE$, construct chord HJ such that HJ touches $\circ BFG$ at G. We claim that $DE = HJ$.

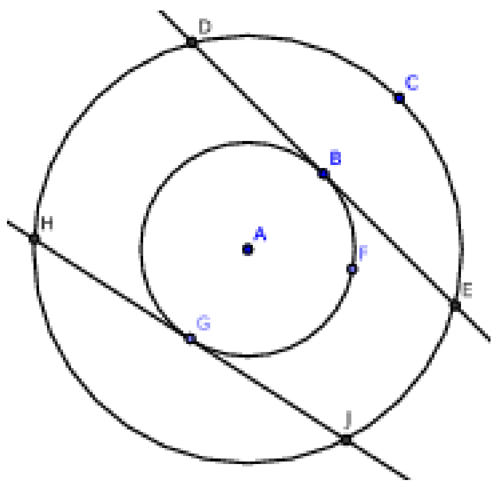

FIGURE 3.0.5. [3.16, #1]

Notice that $AG = AB$ since each are radii of $\circ BFG$. By [3.16], DE and HJ have no other points of intersection with $\circ BFG$. Hence, DE and HJ are equal distance from the center of $\circ BFG$ and also of $\circ DCE$. By [3.14], $DE = HJ$. □

[3.17] Exercises

3. If a parallelogram is circumscribed to a circle, it must be a lozenge, and its diagonals intersect at the center.

PROOF. Construct circle $\circ GHI$ with center E and circumscribe parallelogram $\boxdot BCDA$ to $\circ GHI$ where $\boxdot BCDA$ touches $\circ GHI$ at F, G, H, and I. We claim that $\boxdot BCDA$ is a lozenge and that its diagonals AC and BC intersect E.

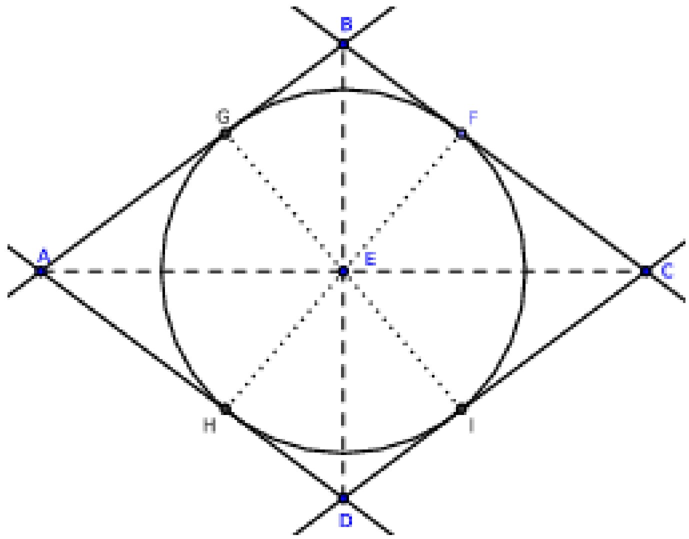

FIGURE 3.0.6. [3.17, #3]

Since $\boxdot BCDA$ is a parallelogram, $EA = EC$ and $EB = ED$. Also, $BC = AD$ and $BA = CD$. By [1.8], $\triangle EAD \cong \triangle ECB$ and $\triangle EBA \cong \triangle EDC$.

Now construct radii EH, EI. Since $\circ GHI$ touches $\boxdot BCDA$ at H and at I, we must have and that $\angle EHD = \angle EID$, since each are tangents to $\circ GHI$ and so are right angles; therefore, $\triangle EHD$ and $\triangle EID$ are right triangles. Since $\triangle EHD$ and $\triangle EID$ share side ED and $EH = EI$ (since both are radii of $\circ GHI$), by [1.26, #6] we have that $\triangle EHD \cong \triangle EID$. Similarly, $\triangle EHA \cong \triangle EIC$, and so it follows that $\triangle EAD \cong \triangle ECD$. It follows that

$$AD = CD = BA = BC$$

and so $\boxdot BCDA$ is a lozenge.

Suppose E is not the center of $\circ GHI$. But E is a point within a circle from which three or more equal segments can be constructed to the circumference, a contradiction [3.9]. Hence, the diagonals of $\boxdot BCDA$ intersect E. $\square$

[3.30] Exercises

1. Suppose that $ABCD$ is a semicircle whose diameter is AD and that the chord BC when extended meets AE (where AE is the extension of AD). Prove that if CE is equal in length to the radius, the arc $AB = 3 \cdot CD$.

PROOF. Suppose $ABCD$ is a semicircle where BC is a chord such that if BC and AD are extended, they intersect at E, and that $CE = DF$. We wish to show that arc $AB = 3 \cdot CD$.

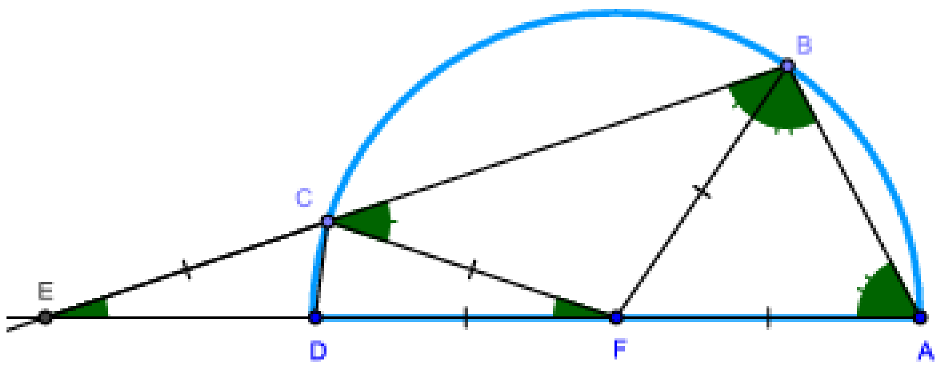

FIGURE 3.0.7. [3.30, #1]

Connect CD, CF, CB, FB, and AB. Notice that

$$CE = DF = CF = BF = AF$$

BY [1.5], we have that $\angle CEF = \angle CFE$, $\angle FCB = \angle FBC$, and $\angle FBA = \angle FAB$.

We will solve this problem using linear algebra. Let

$$\begin{aligned} \angle CEF &= a \\ \angle ECF &= b \\ \angle FCB &= c \\ \angle CFB &= d \\ \angle FBA &= e \\ \angle BFA &= f \end{aligned}$$

Then we have that

$$\begin{aligned} b + c &= 180 \\ 2a + b &= 180 \\ 2c + d &= 180 \\ 2e + f &= 180 \\ a + d + f &= 180 \\ a + c + 2e &= 180 \end{aligned}$$

In matrix form, this is

$$\left[\begin{array}{cccccc|c} 0 & 1 & 1 & 0 & 0 & 0 & 180 \\ 2 & 1 & 0 & 0 & 0 & 0 & 180 \\ 0 & 0 & 2 & 1 & 0 & 0 & 180 \\ 0 & 0 & 0 & 0 & 2 & 1 & 180 \\ 1 & 0 & 0 & 1 & 0 & 1 & 180 \\ 1 & 0 & 1 & 0 & 2 & 0 & 180 \end{array}\right]$$

This matrix in reduced row echelon form is:

$$\left[\begin{array}{cccccc|l} 1 & 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 0 & 0 & 140 \\ 0 & 0 & 1 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

Or,

$$\begin{aligned} \angle CEF &= 20^\circ \\ \angle ECF &= 140^\circ \\ \angle FCB &= 40^\circ \\ \angle CFB &= 100^\circ \\ \angle FBA &= 60^\circ \\ \angle BFA &= 60^\circ \end{aligned}$$

Since $\angle CFD = \angle CFE = \angle CEF$, $\angle BFA = 3 \cdot \angle CFD$. By [7.29, Cor. 1], it follows that $AB = 3 \cdot CD$. □

CHAPTER 4

Solutions: Inscription & Circumscription

[4.4] Exercises

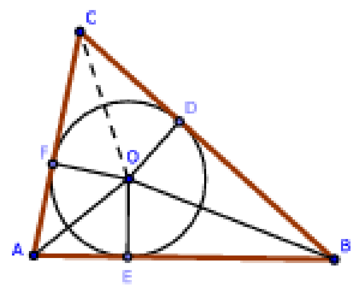

FIGURE 4.0.1. [4.4, #1]

1. In [4.4]: if the points O, C are joined, prove that the angle $\angle ACB$ is bisected. Hence, we prove the existence of the **incenter** of a triangle.

PROOF. Consider $\triangle OFC$ and $\triangle ODC$. By the proof of [4.4], we have that $\angle OFC = \angle ODC$ since each are right; $OF = OD$ since each are radii of $\circ DEF$; and each shares side OC. By [1.26, #6], $\triangle OFC \cong \triangle ODC$, and so $\angle OCF = \angle OCD$. Hence, $\angle ACB$ is bisected, and O is the incenter of $\triangle ABC$. $\square$

[4.5] Exercises

1. The three altitudes of a triangle ($\triangle ABC$) are concurrent. (This proves the existence of the **orthocenter** of a triangle.)

PROOF. Euler's proof: consider a triangle $\triangle ABC$ with circumcenter O and centroid G (i.e., the point of intersection of the medians of the triangle).

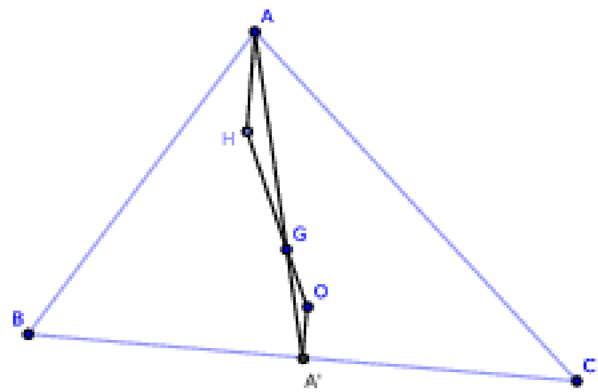

FIGURE 4.0.2. [4.5, #1]

Let A' be the midpoint of BC. Let H be the point such that G is between H and O and $HG = 2 \cdot GO$. Then the triangles $\triangle AGH$, $\triangle A'GO$ are similar by side-angle-side similarity. It follows that $AH \parallel OA'$ and therefore $AH \perp BC$; i.e., it is the altitude from A. Similarly, BH, CH, are the altitudes from B, C. Hence all the altitudes pass through H.[1] $\square$

[1] http://www.artofproblemsolving.com/Wiki/index.php/Orthocenter